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5-field terms in the open superstring effective action

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Abstract: Some time ago the bosonic and fermionic 4-field terms of the non-abelian low energy effective action of the open superstring were obtained, to all order in α'. This was done at tree level by directly generalizing the abelian case, treated some time before, and considering the known expressions of all massless superstring 4-point amplitudes (at tree level). In the present work we obtain the bosonic 5-field terms of this effective action, to all order in α'. This is done by considering the simplified expression of the superstring 5-point amplitude for massless bosons, obtained some time ago.

Keywords: Superstrings and Heterotic Strings, D-branes.

Contents

JHEP03(2005)055

1. Introduction

It has been known for a long time that String Theory modifies the Yang-Mills and Einstein lagrangians by adding α' terms to them [1]. Since the middle eighties many results of this type were found, among which it was seen that the low energy dynamics of abelian open strings (corresponding to photons) was described by the Born-Infeld lagrangian [2]–[6], as long as the field strength is kept constant. This is an all order in α' result. Some non-abelian α' corrections of the effective lagrangian were also determined by that time using the scattering amplitude approach [3, 7, 8].

After the discovery of D-branes as Ramond-Ramond charged states [9] and that the low energy dynamics of non-abelian open superstrings captured an equivalent description in terms of them [10], a supersymmetric generalization of the abelian Born-Infeld lagrangian, in this context, was constructed [11]–[14]. Besides these results, abelian corrections were obtained in [15]–[17], to all order in α'.

A non-abelian generalization of the Born-Infeld lagrangian, in the context of Superstring Theory, has been proposed in [18] by means of a symmetrized trace prescription. The complete $\mathcal{O}({\alpha'}^4)$ terms of [19, 20, 21] have been written following this prescription for the F^6 terms. It has been seen that, in the open superstring effective lagrangian, the covariant derivative terms are as important as the F^n ones [22, 23]. Also, an interesting proposal about the general structure of the non-abelian Born-Infeld action and its covariant derivative terms has been made in [24] by means of the Seiberg-Witten map [25], but this result has unknown coefficients and explicit expressions in $D = 10$ are not given. So, in this sense, in the non-abelian case, the usually known exact results for the lagrangian terms are strictly perturbative in α'. This has been done by different methods which include the 1-loop effective action for $N = 4$ SYM [26, 27]; deformations of the Yang-Mills lagrangian considering either BPS solutions to it [28, 29, 19] or supersymmetry requirements [30]–[33]; and the scattering amplitude approach [3, 7, 8, 34].

It is generally believed that scattering amplitudes are only used in String Theory to find the *first* α' corrections in the effective lagrangians, but this is not true: it was first seen in [17] that all the α' information of the open superstring 4-point amplitudes, of massless bosons and fermions, can be taken to the effective lagrangian, at least in the abelian case. This was soon generalized to the non-abelian case and to the 4-point effective actions of the NS-NS sector of Closed Superstring Theory [35]. In the present work we go further and find the $D^{2n}F^5$ terms of the open superstring non-abelian effective lagrangian, to all order in α'. This is done by using the 5-point amplitude, first completely calculated in [34][1] and afterwards simplified in [36]. In contrast to the 4-point case, due to the presence of poles in the α' terms of the scattering amplitude, this is a very much more complicated problem to solve.

We have organized this paper in a main body and appendices. These last ones contain, besides conventions, identities and tensors, some lengthy formulas and derivations, which are important, but which would otherwise have turned the main body too long. The structure of the main body of the paper is as follows. In section 2 we give a very brief

[1] In [8] a partial computation of this 5-point amplitude was done.

review about scattering amplitudes and the low energy effective lagrangian inferred from them. In section 3 we review the known structure of the effective lagrangian, as far as 3 and 4-point amplitudes are concerned. In section 4 we review the derivation of the 5-point amplitude (at tree level), since this is the starting point for the present paper, and we confirm that this formula satisfies the usual properties of scattering amplitudes in Open Superstring Theory. We have placed the main result of this work, namely, the explicit $D^{2n}F^5$ terms of the effective lagrangian (to all order in α') in section 5. This section also contains the scattering subamplitude that leads to that lagrangian and some explicit examples of α' terms up to $\mathcal{O}(\alpha'^4)$ order.[2] Finally, section 6 contains a summary and final remarks, including future directions. Throughout this work we have treated the very involved calculations that arise in scattering amplitudes using MATHEMATICA's FEYNCALC 3.5 package.

2. Review of scattering amplitudes and low energy effective lagrangian

The M-point tree level scattering amplitude for massless bosons, in superstring theory, is given by [37]

$$\mathcal{A}^{(M)} = i(2\pi)^{10}\delta(k_1+k_2+\cdots+k_M)\cdot \sum_{j_1,j_2,\ldots,j_M}{}' \operatorname{tr}(\lambda^{a_{j_1}}\lambda^{a_{j_2}}\ldots\lambda^{a_{j_M}})\, A(j_1,j_2,\ldots,j_M)\,, \tag{2.1}$$

where the sum $\sum'$ in the indices $\{j_1, j_2, \ldots, j_M\}$ is done over $non-cyclic$ equivalent permutations of the group $\{1, 2, \ldots, M\}$. The matrices λ^{a_j} are in the adjoint representation of the Lie group[3] and $A(j_1, j_2, \ldots, j_M)$, called *subamplitude*, corresponds to the M-point amplitude of open superstrings which do not carry color indices and which are placed in the ordering $\{j_1, j_2, \ldots, j_M\}$ (modulo cyclic permutations).

Since the present work is based in the determination of the effective lagrangian by means of the known expressions of the scattering amplitudes,[4] in the following subsections we briefly review the 3, 4 and 5-point (tree level) subamplitudes of massless bosons in Open Superstring Theory. From them, an on-shell effective lagrangian of the form

$$\mathcal{L}_{\text{ef}} = \mathcal{L}_{YM} + \mathcal{L}_{D^{2n}F^4} + \mathcal{L}_{D^{2n}F^5} \tag{2.2}$$

emerges, where

$$\mathcal{L}_{YM} = -\frac{1}{4}\operatorname{tr}(F^2) \tag{2.3}$$

$$\mathcal{L}_{D^{2n}F^4} = \alpha'^2 g^2 \operatorname{tr}(F^4) + \alpha'^3 g^2 \operatorname{tr}(D^2F^4) + \alpha'^4 g^2 \operatorname{tr}(D^4F^4) + \cdots \tag{2.4}$$

[2] In principle, with the α' expansions that we give in appendix C we could explicitly write the $D^{2n}F^5$ terms up to $\mathcal{O}(\alpha'^6)$ order.

[3] At tree level, the Lie groups SO(N) and USp(N) have been shown to be consistent with the description of superstring interactions[37]. It has also been seen that $U(N)$ is the appropriate gauge group when describing the low energy dynamics of N coincident D-branes [10]. In the present work it will not be necessary to make any especial reference to any of these Lie groups.

[4] To the knowledge of the authors, this method was first introduced in [1].

JHEP03(2005)055

$$\mathcal{L}_{D^{2n}F^5} = \alpha'^3 g^3\, \mathrm{tr}(F^5) + \alpha'^4 g^3\, \mathrm{tr}(D^2F^5) + \alpha'^5 g^3\, \mathrm{tr}(D^4F^5) + \cdots . \tag{2.5}$$

The lagrangian in (2.2) agrees with the open superstring effective lagrangian up to 5-field terms: any difference between them is sensible only to 6 or higher-point scattering amplitudes. Also, there is no unique way in writing its terms since some freedom arises from the $[D,D]F = -ig[F,F]$ relation, the Bianchi identity and integration by parts (see [3, 29] and [27], for example), thus, allowing to interchange some terms from $\mathcal{L}_{D^{2n}F^4}$ and $\mathcal{L}_{D^{2n}F^5}$. In the next section we will specify the convention that we will use in writing the $D^{2n}F^4$ terms.

3. 3 and 4-point subamplitudes and their effective lagrangian in Open Superstring Theory

The tree level 3 and 4-point subamplitudes for massless bosons, in Open Superstring Theory, have been known for a long time. Their on-shell expressions are the following [37], respectively:

$$A(1,2,3) = 2g\left[(\zeta_1\cdot k_2)(\zeta_2\cdot\zeta_3) + (\zeta_2\cdot k_3)(\zeta_3\cdot\zeta_1) + (\zeta_3\cdot k_1)(\zeta_1\cdot\zeta_2)\right], \tag{3.1}$$

$$A(1,2,3,4) = 8\ g^2\ \alpha'^2 \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1-\alpha's-\alpha't)} K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4), \tag{3.2}$$

where

$$K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4) = t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\zeta^1_{\mu_1}k^1_{\nu_1}\zeta^2_{\mu_2}k^2_{\nu_2}\zeta^3_{\mu_3}k^3_{\nu_3}\zeta^4_{\mu_4}k^4_{\nu_4} \tag{3.3}$$

is a kinematic factor, $t_{(8)}$ being a known tensor [37]. The s and t variables in (3.2) are part of the three Mandelstam variables, which are defined as

$$s = -(k_1+k_2)^2, \qquad t = -(k_1+k_4)^2, \qquad u = -(k_1+k_3)^2. \tag{3.4}$$

The Gamma factor in (3.2) has a completely known α' expansion (see appendix C.1) which begins like

$$\alpha'^2\frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1-\alpha's-\alpha't)} = \frac{1}{st} - \frac{\pi^2}{6}\alpha'^2 + \mathcal{O}\left(\alpha'^3\right). \tag{3.5}$$

Since the leading term in the α' expansion of $A(1,2,3,4)$ in (3.2) is the Yang-Mills 4-point subamplitude, $A_{YM}(1,2,3,4)$, then it may also be written as

$$A(1,2,3,4) = \alpha'^2 st\ \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1-\alpha's-\alpha't)} A_{YM}(1,2,3,4). \tag{3.6}$$

This last formula will be useful in the next section, when making a comparison between the 4 and 5-point subamplitudes.

From the expression of $A(1,2,3)$ in (3.1) it is immediate that the effective lagrangian may be written with no $D^{2n}F^3$ terms (as it has already been done in (2.2)) since this amplitude contains no α' corrections and agrees with the one from the Yang-Mills lagrangian.

JHEP03(2005)055

Using the expression of $A(1,2,3,4)$ in (3.2) the lagrangian $\mathcal{L}_{D^{2n}F^4}$ has been determined some time ago [35],[5] to all order in α'. This was done by directly generalizing the procedure considered in [17] for the non-abelian case. The final result is the following:[6]

$$\begin{aligned}\mathcal{L}_{D^{2n}F^4} = & -\frac{1}{8}g^2\alpha'^2 \int\!\!\int\!\!\int\!\!\int \left\{ \prod_{j=1}^{4} d^{10}x_j \; \delta^{(10)}(x-x_j) \right\} \times \\ & \times f_{\rm sym}\left(\frac{(D_1+D_2)^2+(D_3+D_4)^2}{2}, \frac{(D_1+D_4)^2+(D_2+D_3)^2}{2} \right) \times \\ & \times t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\, {\rm tr}\left(F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)\right), \end{aligned} \tag{3.7}$$

where the function f is given by

$$f(s,t) = \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} - \frac{1}{\alpha'^2 st}\,. \tag{3.8}$$

In (3.7) '$f_{\rm sym}$' denotes that in the power series of it, which involves powers of s and t, we are using a symmetrized convention:

$$(s^k t^l)_{\rm sym} = \frac{(\text{ Sum of all different permutations of k powers of } s \text{ and l powers of } t\text{ })}{\begin{pmatrix} k+l \\ k \end{pmatrix}}. \tag{3.9}$$

In the case of ordinary numbers the formula in (3.9) coincides with $s^k t^l$, but this does not happen in the case of covariant derivative operators, as it is in (3.7).

The terms of $\mathcal{L}_{D^{2n}F^4}$ were written explicitly up to $\mathcal{O}(\alpha'^5)$ order in [35] .

The fact that $A(1,2,3,4)$ has no poles at any (non zero) α' order made the construction of $\mathcal{L}_{D^{2n}F^4}$, in (3.7), quite direct: besides some trace factors, the lagrangian is constructed in terms of the 4-point Gamma factor in which the momenta are substituted by covariant derivatives (appropriately symmetrized). All this implies an enormous simplification with respect to other methods of obtaining the α' correction terms in the effective lagrangian, since all of them require, order by order in α', of an explicit construction of gauge invariant terms with unknown coefficients which are afterwards determined by some matching with Superstring Theory 4-point amplitudes (see [3, 23, 29] and [27], for example).

[5] In fact, not only the $D^{2n}F^4$ terms were determined in [35], but also all boson-fermion and fermion-fermion terms which are sensible to 4-point amplitudes.

[6] There are three differences between eq. (3.7) and the corresponding one in [35, section 2.2]:

1. We have now written the F^4 terms of the integrand in an abbreviated manner, by means of the $t_{(8)}$ tensor and, as a consequence of the symmetries of this tensor, it has been possible to write $\mathcal{L}_{D^{2n}F^4}$ in terms of only one function $f_{\rm sym}$, instead of the three ones contained in the $\mathcal{G}_{\alpha'{\rm corr.}}^{\;a_1a_2a_3a_4}(s,t,u)$ function of [35].
2. We have introduced a symmetrized prescription in the α' expansion of function f, explained in eq. (3.9), and that is why we have now called it $f_{\rm sym}$.
3. We have included derivative terms which were not considered in [35], namely, the ones with $D^2 = D^\alpha D_\alpha$ operating on a same field strength.

Neither item 2 nor item 3 invalidate the result of [35], since there were considered results sensible up to 4-point amplitudes only.

JHEP03(2005)055

4. The 5-point amplitude of massless bosons in Open Superstring Theory

A first calculation of the 5-point amplitude for massless bosons in Open Superstring Theory was done a long time ago in [8]. This was a partial calculation with enough information to calculate the F^5 and the D^2F^4 terms that appear at $\mathcal{O}(\alpha'^3)$ order in the effective lagrangian (2.2). Long after this result, in [29] it was seen that, of all these terms, only the D^2F^4 ones were correctly determined. The corrected F^5 terms of [29] were also confirmed subsequently by other methods [30, 34, 27, 33].

Since the main result of the present work, namely, the determination of the $D^{2n}F^5$ terms of the effective lagrangian, lies on the computation of the 5-point subamplitude, in the next subsection we will review the main steps of it in some detail. In the subsection after the next one we will check that our 5-point formula satisfies the usual properties of massless open string subamplitudes [37, 38]: cyclicity, (on-shell) gauge invariance, world-sheet parity and factorizability.

4.1 Review of the 5-point subamplitude derivation

A complete expression for the 5-point amplitude was first obtained in [34]:

$$
\begin{aligned}
A(1,2,3,4,5) =&\\
= 2g^3(2\alpha')^2\Big[&(\zeta_1\cdot\zeta_2)(\zeta_3\cdot\zeta_4)\Big\{(\zeta_5\cdot k_1)(k_2\cdot k_3)L_2 - (\zeta_5\cdot k_2)(k_1\cdot k_3)L_3 +\\
&\qquad + (\zeta_5\cdot k_3)\big((k_2\cdot k_4){L_3}' + (k_2\cdot k_3)L_2\big)\Big\} +\\
&+ (\zeta_1\cdot\zeta_3)(\zeta_2\cdot\zeta_4)\Big\{-(\zeta_5\cdot k_1)(k_2\cdot k_3)L_7 +\\
&\qquad + (\zeta_5\cdot k_2)\big((k_3\cdot k_4){L_1}' - (k_2\cdot k_3)L_7\big) -\\
&\qquad - (\zeta_5\cdot k_3)(k_1\cdot k_2)L_1\Big\} +\\
&+ (\zeta_1\cdot\zeta_4)(\zeta_2\cdot\zeta_3)\Big\{(\zeta_5\cdot k_1)\big((k_3\cdot k_4){K_4}' - (k_1\cdot k_3)K_5\big) -\\
&\qquad - (\zeta_5\cdot k_2)(k_1\cdot k_3)K_5 + (\zeta_5\cdot k_3)(k_1\cdot k_2)K_4\Big\} +\\
&+ (\zeta_2\cdot\zeta_3)\Big\{(\zeta_5\cdot k_1)\big((\zeta_1\cdot k_2)(\zeta_4\cdot k_3)L_2 - (\zeta_1\cdot k_3)(\zeta_4\cdot k_2)L_7\big) +\\
&\qquad + (\zeta_5\cdot k_2)\Big((\zeta_1\cdot k_3)(\zeta_4\cdot k_1)K_5 + (\zeta_1\cdot k_3)(\zeta_4\cdot k_3){L_4}' +\\
&\qquad\qquad + (\zeta_1\cdot k_2)(\zeta_4\cdot k_3)L_2 + (\zeta_1\cdot k_4)(\zeta_4\cdot k_3){K_4}'\Big) -\\
&\qquad - (\zeta_5\cdot k_3)\Big((\zeta_1\cdot k_2)(\zeta_4\cdot k_1)K_4 + (\zeta_1\cdot k_3)(\zeta_4\cdot k_2)L_7 +\\
&\qquad\qquad + (\zeta_1\cdot k_2)(\zeta_4\cdot k_2)L_4 + (\zeta_1\cdot k_4)(\zeta_4\cdot k_2){K_5}'\Big)\Big\} +
\end{aligned}
$$

$$
\begin{aligned}
&+(\zeta_1\cdot\zeta_4)\Big\{(\zeta_5\cdot k_2)\left(((\zeta_2\cdot k_1)(\zeta_3\cdot k_4)K_2-(\zeta_2\cdot k_4)(\zeta_3\cdot k_1)K_3\right)-\\
&\qquad-(\zeta_5\cdot k_1)\Big(((\zeta_2\cdot k_3)(\zeta_3\cdot k_4){K_4}'-(\zeta_2\cdot k_4)(\zeta_3\cdot k_2){K_5}'+\\
&\qquad\qquad+(\zeta_2\cdot k_4)(\zeta_3\cdot k_4){K_1}'-(\zeta_2\cdot k_1)(\zeta_3\cdot k_4)K_2\Big)+\\
&\qquad+(\zeta_5\cdot k_4)\Big(((\zeta_2\cdot k_1)(\zeta_3\cdot k_2)K_4+(\zeta_2\cdot k_1)(\zeta_3\cdot k_1)K_1-\\
&\qquad\qquad-(\zeta_2\cdot k_3)(\zeta_3\cdot k_1)K_5-(\zeta_2\cdot k_4)(\zeta_3\cdot k_1)K_3\Big)\Big\}\Big]+\\
&+(\text{cyclic permutations of indexes }(1,2,3,4,5))\,.
\end{aligned}
\tag{4.1}
$$

In this formula, K_i, ${K_i}'$, L_i and ${L_i}'$ are α' dependent factors,[7] defined by a double integral of the form

$$
\int_0^1 dx_3\int_0^{x_3}dx_2\;x_3^{2\alpha'\alpha_{13}}(1-x_3)^{2\alpha'\alpha_{34}}{x_2}^{2\alpha'\alpha_{12}}(1-x_2)^{2\alpha'\alpha_{24}}(x_3-x_2)^{2\alpha'\alpha_{23}}\cdot\varphi(x_2,x_3)\,, \tag{4.2}
$$

where the function $\varphi(x_2,x_3)$ has a specific expression for each of them (see [34, appendix A.1] for further details). They can all be calculated as a product of a Beta and a Hypergeometric function [8] and they have a well defined Laurent expansion in α' (after regularizing in some cases). For example, in the case of the factors K_2 and K_3 we have that [34]:[8]

$$
\begin{aligned}
K_2 \;=\;& \frac{1}{(2\alpha')^2}\left\{\frac{1}{\alpha_{12}\alpha_{34}}\right\}-\frac{\pi^2}{6}\left\{\frac{\alpha_{51}\alpha_{12}-\alpha_{12}\alpha_{34}+\alpha_{34}\alpha_{45}}{\alpha_{12}\alpha_{34}}\right\}+\\
&+\zeta(3)(2\alpha')\left\{\frac{\alpha_{12}^2\alpha_{51}-\alpha_{34}^2\alpha_{12}+\alpha_{45}^2\alpha_{34}+\alpha_{51}^2\alpha_{12}-\alpha_{12}^2\alpha_{34}+\alpha_{34}^2\alpha_{45}-2\alpha_{12}\alpha_{23}\alpha_{34}}{\alpha_{12}\alpha_{34}}\right\}+\\
&+\mathcal{O}((2\alpha')^2)\,,
\end{aligned}
\tag{4.3}
$$

$$
K_3 \;=\; \frac{\pi^2}{6}-\zeta(3)\;(2\alpha')\left\{\alpha_{12}+\alpha_{23}+\alpha_{34}+\alpha_{45}+\alpha_{51}\right\}+\mathcal{O}((2\alpha')^2)\,. \tag{4.4}
$$

In (4.3) and (4.4) we are using the notation

$$
\alpha_{ij}=k_i\cdot k_j\qquad(i,j=1,2,3,4,5;\;\;i\neq j)\,. \tag{4.5}
$$

In [34, appendix A.2], among some relations, the following were found:

$$
\begin{aligned}
\alpha_{34}K_2 &= \alpha_{13}K_1+\alpha_{23}K_4\\
\alpha_{24}K_3 &= \alpha_{12}K_1-\alpha_{23}K_5\\
\alpha_{13}L_1 &= \alpha_{34}{L_3}'-\alpha_{23}{L_4}'
\end{aligned}
$$

[7]In [34] they were called ‘kinematic factors’ but we will not use this terminology any longer since the word ‘kinematic’ is usually reserved for expressions which depend on both, momenta *and* polarizations. In the present work we will use the terminology ‘α' dependent factor’ to denote an expression which depends on α' *and* the momenta k_i.

[8]In [34], the expansions of K_2 and K_3 contained the variables ρ and α_{24} which we have already substituted using the relations (C.9), (C.12) and (C.14) of appendix C.2.

$$\begin{aligned}
\alpha_{12}K_2 &= \alpha_{24}{K_1}' + \alpha_{23}{K_4}' \\
\alpha_{13}K_3 &= \alpha_{34}{K_1}' - \alpha_{23}{K_5}' \\
\alpha_{24}{L_1}' &= \alpha_{12}L_3 - \alpha_{23}L_4 \\
\alpha_{34}{K_4}' - \alpha_{13}K_5 &= \alpha_{12}K_4 - \alpha_{24}{K_5}' \,.
\end{aligned} \tag{4.6}$$

The first three of them were explicitly written in [34, eq. (A.6)]. The following three relations can be obtained from the first ones by the *duality* operation mentioned in that appendix.[9] All of these expressions can be derived using the definition of each K_i, ${K_i}'$, L_i and ${L_i}'$, as a double integral (see [34, appendix A.1]), and integration by parts. The last relation in (4.6) comes from demanding invariance of K_6 under the mentioned *duality* operation and using its expression given in appendix A.2 of the same reference.

Now, in [36] there were found additional relations, independent of the ones in (4.6):

$$\begin{aligned}
K_2 + \frac{(K_1 - L_3 + {K_1}' - {L_3}')}{2} &= 0\,, \\
K_1 - K_4 + K_5 &= 0\,, \\
{K_1}' - {K_4}' + {K_5}' &= 0\,, \\
K_5 + {K_5}' + \frac{(K_1 - L_4 + {K_1}' - {L_4}')}{2} &= 0\,, \\
L_2 - L_3 - {L_4}' &= 0\,, \\
L_2 - {L_3}' - L_4 &= 0\,, \\
K_1 - {K_1}' - (L_4 - {L_4}') &= 0\,.
\end{aligned} \tag{4.7}$$

These independent relations come from the very definition of each factor, since the function $\varphi(x_2, x_3)$ in every case is a fraction. For example, for the factors K_1, K_4 and K_5, the corresponding $\varphi(x_2, x_3)$ function is given by

$$\varphi_{K_1}(x_2, x_3) = \frac{1}{x_2\ x_3}\,, \quad \varphi_{K_4}(x_2, x_3) = \frac{1}{x_2\ (x_3 - x_2)}\,, \qquad \varphi_{K_5}(x_2, x_3) = \frac{1}{x_3\ (x_3 - x_2)}\,, \tag{4.8}$$

which can easily be seen to satisfy

$$\varphi_{K_1}(x_2, x_3) - \varphi_{K_4}(x_2, x_3) + \varphi_{K_5}(x_2, x_3) = 0\,. \tag{4.9}$$

The integrated version of eq. (4.9) is precisely the second of the equations in (4.7).

Besides all these relations, we now introduce a new factor, T, defined as

$$T = (2\alpha')^2 \left[\alpha_{12}\alpha_{34}\ K_2 + (\alpha_{51}\ \alpha_{12} - \alpha_{12}\alpha_{34} + \alpha_{34}\ \alpha_{45})K_3\right]. \tag{4.10}$$

The same as K_3, this factor remains invariant under cyclic permutations of indexes $(1, 2, 3, 4, 5)$. We prove this in appendix C.2.

The factors K_2 and K_3 have been written in terms of Beta and Hypergeometric functions in equations (C.10) and (C.11).

[9] From the point of view of the string world-sheet, this *duality* operation is nothing else than a *twist* of a disk with five insertions, with respect to the fifth vertex. This will be seen in the third item of subsection 4.2.

So, summarizing, the 5-point amplitude in (4.1) is given in terms of sixteen factors K_i, ${K_i}'$, L_i and ${L_i}'$, which are related with the additional factor T by fifteen independent linear relations given in (4.6), (4.7) and (4.10). This allows to write the 5-point amplitude in (4.1) in terms of only two factors, which we choose to be T and K_3[10]

$$A(1,2,3,4,5) = T \cdot A(\zeta, k) + (2\alpha')^2 K_3 \cdot B(\zeta, k)\,. \tag{4.11}$$

Here, $A(\zeta, k)$ and $B(\zeta, k)$ are two kinematical expressions which are known explicitly, after all the substitutions of the factors have been done in (4.1). They can be identified with subamplitudes of specific terms of the effective lagrangian (2.2), as we will see in the next lines.

In appendix C.2 we have that the α' expansion of T begins as

$$T = 1 + \mathcal{O}((2\alpha')^3)\,. \tag{4.12}$$

Substituting the leading terms of the α' expansions of T and $(2\alpha')^2 K_3$ in (4.11), we have that $A(\zeta, k)$ should agree with the Yang-Mills 5-point subamplitude and that $B(\zeta, k)$ should agree with the F^4 terms 5-point subamplitude. We have checked (computationally) that this really happens on-shell, after using momentum conservation and physical state conditions.

So our final formula for the 5-point subamplitude is

$$A(1,2,3,4,5) = T \cdot A_{YM}(1,2,3,4,5) + (2\alpha')^2 K_3 \cdot A_{F^4}(1,2,3,4,5)\,. \tag{4.13}$$

In (D.1) and (4.14) we give the expressions for $A_{YM}(1,2,3,4,5)$ and $A_{F^4}(1,2,3,4,5)$, respectively.

The formula in (4.13) has exactly the same structure of the corresponding one for the 4-point subamplitude, written in (3.6), but with two kinematic expressions and two factors which contain the α' dependence.

4.2 Properties satisfied by the 5-point amplitude

On this subsection we confirm that $A(1,2,3,4,5)$, given in (4.13), satisfies the usual properties of massless open string subamplitudes, namely, cyclic invariance, on-shell gauge invariance, world-sheet parity and factorizability.

1. **Cyclic invariance.** It was mentioned in the previous subsection, and it is proved in appendix C.2, that the factors T and K_3 are invariant under cyclic permutations of indexes $(1,2,3,4,5)$. Now, once $A_{YM}(1,2,3,4,5)$ and $A_{F^4}(1,2,3,4,5)$ are the Yang-Mills and the F^4 terms 5-point subamplitudes, which are non-abelian field theory amplitudes, by construction they are cyclic invariant. So, in this sense, the cyclic invariance of $A(1,2,3,4,5)$ is already manifestly written in formula (4.13).

[10]Eq. (4.11) is the same as the one in Table 2 of [36], but with not exactly the same choice of momentum dependent factors and kinematic expressions.

2. **On-shell gauge invariance.** This property consists in that the subamplitude should become zero if any of the polarizations ζ_i is substituted by the corresponding momentum k_i, after using physical state ($\zeta_j \cdot k_j = 0$) and on-shell ($k_j^2 = 0$) conditions for all external string states, together with momentum conservation [37]. That this indeed happens in (4.13) can be understood from the fact that $A_{YM}(1,2,3,4,5)$ and $A_{F^4}(1,2,3,4,5)$ are 5-point subamplitudes that come from gauge invariant terms, so both of them should independently become zero when doing $\zeta_i \to k_i$ for any $i = 1,2,3,4,5$. So the on-shell gauge invariance of $A(1,2,3,4,5)$ is also manifestly written in (4.13).

   In any case, the explicit check of the on-shell gauge invariance of $A_{F^4}(1,2,3,4,5)$ can be seen as follows. This subamplitude is given by

$$\begin{aligned} A_{F^4}(1,2,3,4,5) = 2g^3 \Bigg\{ \Bigg[& K(\zeta_1,\zeta_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5) + \frac{1}{\alpha_{12}} \times \\ & \times \Bigg((\zeta_1 \cdot \zeta_2) K(k_1,k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5) + \\ & \quad + (\zeta_1 \cdot k_2) K(\zeta_2,k_1+k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5) - \\ & \quad - (\zeta_2 \cdot k_1) K(\zeta_1,k_1+k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5) \Bigg) \Bigg] + \\ & + (\text{cyclic permutations}) \Bigg\} . \end{aligned} \tag{4.14}$$

   It has been obtained as the coefficient of $\pi^2/6\ (2\alpha')^2$ in the α' series of the subamplitude $A_{D^{2n}F^4}(1,2,3,4,5)$, given in (E.1). As mentioned in appendix E, in this formula the expression $K(A,a;B,b;C,c;D,d)$ denotes the same kinematic construction of (3.3), evaluated in the corresponding variables. Due to the symmetries of the $t_{(8)}$ tensor (see appendix B.1), it is not difficult to see that this last expression becomes zero whenever any of the ζ_i is substituted by k_i.

3. **World-sheet parity.** The tree level interaction of n open strings is described by a conformal field theory on a disk with n insertions on its boundary [39]. A world-sheet parity transformation, $\sigma \to l - \sigma$ (where $\sigma\ \varepsilon\ [0,\ l]$ is the string internal coordinate), corresponds to a twisting of this disk with respect to any of those insertions (in figure 1, for example, it is shown a twisting with respect to the fifth insertion). When there is an interaction of an odd number of open strings, the subamplitude changes sign under a world-sheet parity transformation [37, 39] . So, in the case of five interacting superstrings, a twisting with respect to, say the fifth one,

$$(1,2,3,4,5) \to (4,3,2,1,5)\,, \tag{4.15}$$

   should imply that

$$A(1,2,3,4,5) = -A(4,3,2,1,5)\,. \tag{4.16}$$

   To see that the amplitude in (4.13) indeed satisfies this condition it can be argued as follows. First, in appendix C.2.2 it is proved that the factors T and K_3 are invariant under a twisting transformation. Second, the Yang-Mills subamplitude

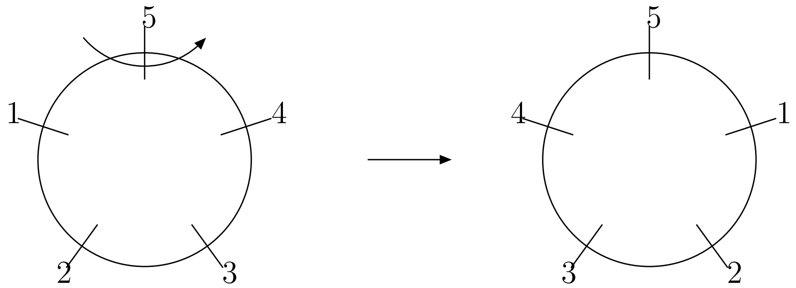

Figure 1: Twisting of a disk with five insertions, with respect to the fifth one. Under this operation the indexes $(1,2,3,4,5)$ transform as indicated in equation (4.15).

$A_{YM}(1,2,3,4,5)$ already satisfies (4.16) [38].[11] And third, using the symmetries of the $t_{(8)}$ tensor it is not difficult to prove that $A_{F^4}(1,2,3,4,5)$, in (4.14), does also satisfy the twisting condition (4.16).

4. **Factorizability.** There are two non trivial tests that confirm the right structure of the poles of $A(1,2,3,4,5)$ in (4.13). The factorization comes when a particular limit is considered. In that limit the subamplitude diverges with simple poles and its residues factorize in terms of the 4-point subamplitude.

   Here we follow very closely [38, section 6].

   - **Soft boson factorization.** This corresponds to the case when one of the bosons, say the fifth one, becomes soft ($k_5 \rightarrow 0$). It is very well known (see [38], for example) that the n-point Yang-Mills subamplitude satisfies the factorization mentioned here. In the case $n=5$ it has the following form:

$$A_{YM}(1,2,3,4,5)_{k_5\rightarrow 0} \sim \ g \left(\frac{\zeta_5 \cdot k_1}{k_5 \cdot k_1} - \frac{\zeta_5 \cdot k_4}{k_5 \cdot k_4} \right) \ A_{YM}(1,2,3,4)\,. \tag{4.17}$$

     Also, taking $k_5 \rightarrow 0$ in (4.14) it is not difficult to see that $A_{F^4}(1,2,3,4,5)$ satisfies exactly the same factorization. Therefore, taking the $k_5 \rightarrow 0$ limit in (4.13) and using that

$$\begin{aligned} A_{YM}(1,2,3,4) &= 2\ g^2 \frac{K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4)}{\alpha_{12}\alpha_{14}}\,, \\ A_{F^4}(1,2,3,4) &= -\ 2\ g^2\ K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4)\,, \end{aligned} \tag{4.18}$$

     we have that

$$\begin{aligned} A(1,2,3,4,5)_{k_5\rightarrow 0} \sim 2g^3 \left(\frac{\zeta_5 \cdot k_1}{k_5 \cdot k_1} - \frac{\zeta_5 \cdot k_4}{k_5 \cdot k_4} \right) \left(\frac{1}{\alpha_{12}\alpha_{14}} T - (2\alpha')^2 K_3 \right) \Big|_{k_5=0} \times \\ \times K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4)\,. \end{aligned} \tag{4.19}$$

[11] This twisting transformation is also called 'inversion' in [38] since due to the cyclic symmetry, (4.16) can also be written as $A(1,2,3,4,5) = -A(5,4,3,2,1)$.

Now, if $k_5 = 0$ this implies that $\alpha_{45} = \alpha_{51} = 0$ and $\alpha_{14} = \alpha_{23}$ (see eq. (C.13)), so we have that

$$\begin{aligned} A(1,2,3,4,5)_{k_5\to 0} \sim 2g^3 \left(\frac{\zeta_5\cdot k_1}{k_5\cdot k_1} - \frac{\zeta_5\cdot k_4}{k_5\cdot k_4}\right) \times \\ \times \left(\frac{1}{\alpha_{12}\alpha_{23}}T \; - \; (2\alpha')^2 K_3\right)\Bigg|_{\alpha_{45}=\alpha_{51}=0} \times \\ \times K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4)\,. \end{aligned} \tag{4.20}$$

In this last relation, the expression evaluated in $\alpha_{45} = \alpha_{51} = 0$ is basically the Gamma factor of the 4-point amplitude evaluated in α_{12} and α_{23} (see eq. (C.17)), which finally allows us to write

$$A(1,2,3,4,5)_{k_5\to 0} \sim \; g\left(\frac{\zeta_5\cdot k_1}{k_5\cdot k_1} - \frac{\zeta_5\cdot k_4}{k_5\cdot k_4}\right) \; A(1,2,3,4)\,. \tag{4.21}$$

- **Factorization of collinear poles.** This corresponds to the case of two consecutive bosons, say the first and the second one, that become parallel. The Yang-Mills subamplitude factorization, in this case, has the following form [38]:

$$\begin{aligned} & A_{YM}(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5)_{k_1||k_2} \sim \\ & \sim \frac{1}{2(k_1\cdot k_2)} \; V^\mu \frac{\partial}{\partial\zeta^\mu} A_{YM}(\zeta,k_1+k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5)\,, \end{aligned} \tag{4.22}$$

where

$$V^\mu = -g\Big[(\zeta_1\cdot\zeta_2)(k_1-k_2)^\mu - 2(\zeta_2\cdot k_1){\zeta_1}^\mu + 2(\zeta_1\cdot k_2){\zeta_2}^\mu\Big] \tag{4.23}$$

comes from the Yang-Mills 3-point vertex given in (D.2):

$$V_\mu = -i \; g \; V^{(3)}_{YM \;\; \mu_1\mu_2\mu}(k_1,k_2,-k_1-k_2) \; \zeta_1^{\mu_1}\zeta_2^{\mu_2}\,. \tag{4.24}$$

Notice that the subamplitude on the left handside of (4.22) is a 5-point subamplitude while the one on the right handside is a 4-point one.
Now, when considering the $k_1||k_2$ limit in (4.14) the dominant term is precisely the one which has the denominator α_{12} (recall that $k_1\cdot k_2 = 0$ in this limit, due to the on-shell condition). The three terms in the residue of α_{12} in (4.14) can be seen to match with the corresponding ones of V^μ, when substituted in (4.23), so $A_{F^4}(1,2,3,4,5)$ also satisfies the factorization in (4.22).
So, in the same way as was done in the previous item, taking the $k_1||k_2$ limit in (4.13) leads to

$$\begin{aligned} & A(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5)_{k_1||k_2} \sim \\ & \sim \frac{1}{2(k_1\cdot k_2)} \; V^\mu \frac{\partial}{\partial\zeta^\mu}\Big[2\;g^2\;K(\zeta,k_1+k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5)\Big] \times \\ & \times \left(\frac{T}{[(k_1+k_2)\cdot k_3]\;[(k_1+k_2)\cdot k_5]} \; - \; (2\alpha')^2 K_3\right)\Bigg|_{\alpha_{12}=0}\,. \end{aligned} \tag{4.25}$$

Using relations (C.12) and (C.15) it can easily be seen that $(k_1 + k_2) \cdot k_3 = \alpha_{45}$ and $(k_1 + k_2) \cdot k_5 = \alpha_{34}$ when $\alpha_{12} = 0$, so the factor in the third line in (4.25) becomes, once more, the Gamma factor of the 4-point subamplitude (see (C.17)). So we finally have that:

$$
\begin{aligned}
&A(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5)_{k_1 || k_2} \sim \\
&\quad \sim \frac{1}{2(k_1 \cdot k_2)} \, V^\mu \frac{\partial}{\partial \zeta^\mu} A(\zeta, k_1 + k_2; \zeta_3, k_3; \zeta_4, k_4; \zeta_5, k_5) \, .
\end{aligned} \tag{4.26}
$$

5. 5-field terms in the low energy effective lagrangian

In the case of the $D^{2n}F^4$ terms lagrangian, $\mathcal{L}_{D^{2n}F^4}$, given in (3.7), its determination was quite direct from the corresponding scattering subamplitude (3.2). Unfortunately, the same procedure does not work in the case of $\mathcal{L}_{D^{2n}F^5}$: for each of the two terms in the 5-point subamplitude in (4.13), there does not exist a local lagrangian, at each order in α', which reproduces the corresponding subamplitude.[12] There only exists an effective lagrangian which reproduces the *sum* of the two terms in (4.13). This can be argued by noticing that the α' expansion for each of those terms does not satisfy the factorizability property but, as it was seen in (4.21) and (4.26), this property is indeed satisfied if *both* terms are included.

In this section we deal with the main result of this work, namely, the 5-field terms in the effective lagrangian. By this we mean the explicit determination of the $D^{2n}F^5$ terms[13] by means of the 5-point subamplitude (4.13). These terms do not contribute to the *abelian* effective lagrangian: in the abelian limit all of them become zero[14].

The determination of the $D^{2n}F^5$ terms, by means of the complete 5-point subamplitude, is by far more complicated than the corresponding one of the $D^{2n}F^4$ terms [35] given in (3.7). This happens because, at every order in α', say α'^{k+3} $(k = 0, 1, 2, \ldots)$, the 5-point subamplitude receives contributions not only from the $D^{2k}F^5$ terms, but also from the $D^{2k+2}F^4$ ones. The contribution of the first type of terms has no poles, but the contribution of the second type of terms contains poles and also regular terms.

In the next subsection we first write the final expression for the $D^{2n}F^5$ terms scattering subamplitude (and leave the details of its derivation to appendix F). In the second subsection we write the desired lagrangian terms in short notation (using tensors and α' dependent functions) and in the last one we give explicit examples of the $D^{2n}F^5$ and the $D^{2n}F^4$ terms up to $\mathcal{O}(\alpha'^4)$ order.

5.1 The $D^{2n}F^5$ terms 5-point subamplitude

From (2.2) we have that the contribution of the $D^{2n}F^5$ terms to the open superstring

[12]Only for the first term in (4.13), at order 0 in α', it is possible to find a local lagrangian, namely, the Yang-Mills lagrangian.

[13]The $D^{2n}F^4$ terms also contain 5-field terms, but in the present context we refer to those which explicitly contain 5 field strengths and covariant derivatives of them.

[14]In the case of an odd number of strings, the abelian limit of the scattering amplitude (2.1) gives a null result and, therefore, no interaction term for them goes in the effective lagrangian. This happens due to the world-sheet parity antisymmetry, (4.16), of the string subamplitudes.

5-point subamplitude is obtained as

$$A_{D^{2n}F^5}(1,2,3,4,5) = A(1,2,3,4,5) - A_{YM}(1,2,3,4,5) - A_{D^{2n}F^4}(1,2,3,4,5)\,, \qquad (5.1)$$

where the $D^{2n}F^4$ terms 5-point subamplitude, $A_{D^{2n}F^4}(1,2,3,4,5)$, is given in (E.1). In appendix F we have worked with the expression in (5.1), checking out that all poles cancel (as expected), arriving to the following expression:

$$\begin{aligned} A_{D^{2n}F^5}(1,2,3,4,5) = g^3 \Bigg\{ \Bigg[& H^{(1)} \cdot h^{(1)}(\zeta,k) + P^{(1)} \cdot p^{(1)}(\zeta,k) + \\ & + U^{(1)} \cdot u^{(1)}(\zeta,k) + W^{(1)} \cdot w^{(1)}(\zeta,k) + Z^{(1)} \cdot z^{(1)}(\zeta,k) \Bigg] + \\ & + (\text{cyclic permutations}) \Bigg\} + g^3\ \Delta \cdot \delta(\zeta,k)\,, \end{aligned} \qquad (5.2)$$

where $H^{(1)}$, $P^{(1)}$, $U^{(1)}$, $W^{(1)}$, $Z^{(1)}$ and Δ are α' dependent factors (obtained from the Gamma, the T and the K_3 factors) which are given in appendix C.3, while $h^{(1)}(\zeta,k)$, $p^{(1)}(\zeta,k)$, $u^{(1)}(\zeta,k)$, $w^{(1)}(\zeta,k)$, $z^{(1)}(\zeta,k)$ and $\delta(\zeta,k)$ are kinematical expressions (with no poles) which depend on the polarizations ζ_i and momenta k_i $(i=1,2,3,4,5)$. There is no summation over cyclic permutations of the term '$g^3\ \Delta\cdot\delta(\zeta,k)$' because Δ and $\delta(\zeta,k)$ are already cyclic invariant expressions.

The kinematical expressions in (5.2) are given by

$$h^{(1)}(\zeta,k) = t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \zeta^1_{\mu_1} k^1_{\nu_1} \zeta^2_{\mu_2} k^2_{\nu_2} \zeta^3_{\mu_3} k^3_{\nu_3} \zeta^4_{\mu_4} k^4_{\nu_4} \zeta^5_{\mu_5} k^5_{\nu_5}\,, \qquad (5.3)$$

$$p^{(1)}(\zeta,k) = 2\,(\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \zeta^1_{\mu_1} k^1_{\nu_1} \zeta^2_{\mu_2} k^2_{\nu_2} \zeta^3_{\mu_3} k^3_{\nu_3} \zeta^4_{\mu_4} k^4_{\nu_4} \zeta^5_{\mu_5} k^5_{\nu_5}\,, \qquad (5.4)$$

$$\begin{aligned} u^{(1)}(\zeta,k) = & \Bigg[\frac{1}{2} t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \alpha_{34} + 2\ t_{(8)}^{\mu_4\nu_4\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2} k_1^{[\mu_3} k_2^{\nu_3]} + \\ & + 2 t_{(8)}^{\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5\mu_1\nu_1} k_1^{[\mu_2} k_3^{\nu_2]} - \\ & - 2\ t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} k_1^{[\mu_5} k_4^{\nu_5]} - 2\ t_{(8)}^{\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} k_1^{[\mu_4} k_5^{\nu_4]} \Bigg] \times \\ & \times \zeta^1_{\mu_1} k^1_{\nu_1} \zeta^2_{\mu_2} k^2_{\nu_2} \zeta^3_{\mu_3} k^3_{\nu_3} \zeta^4_{\mu_4} k^4_{\nu_4} \zeta^5_{\mu_5} k^5_{\nu_5}\,, \end{aligned} \qquad (5.5)$$

$$w^{(1)}(\zeta,k) = 4\ t_{(8)}^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \zeta^1_{\mu_1} k^1_{\nu_1} \zeta^2_{\mu_2} k^2_{\nu_2} \zeta^3_{\mu_3} k^3_{\nu_3} \zeta^4_{\mu_4} k^4_{\nu_4} \zeta^5_{\mu_5} k^5_{\nu_5} k_2^{[\mu_1} k_5^{\nu_1]}\,, \qquad (5.6)$$

$$z^{(1)}(\zeta,k) = 4\ t_{(8)}^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \zeta^1_{\mu_1} k^1_{\nu_1} \zeta^2_{\mu_2} k^2_{\nu_2} \zeta^3_{\mu_3} k^3_{\nu_3} \zeta^4_{\mu_4} k^4_{\nu_4} \zeta^5_{\mu_5} k^5_{\nu_5} \left(k_3^{[\mu_1} k_4^{\nu_1]} - k_2^{[\mu_1} k_5^{\nu_1]} \right) \qquad (5.7)$$

$$\begin{aligned} \delta(\zeta,k) = & \frac{1}{5} \Bigg[t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \alpha_{51}\alpha_{12} + 4(\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \alpha_{23}\alpha_{45} \Bigg] \times \\ & \times \zeta^1_{\mu_1} k^1_{\nu_1} \zeta^2_{\mu_2} k^2_{\nu_2} \zeta^3_{\mu_3} k^3_{\nu_3} \zeta^4_{\mu_4} k^4_{\nu_4} \zeta^5_{\mu_5} k^5_{\nu_5} + (\text{cyclic permutations})\,. \end{aligned} \qquad (5.8)$$

The expression for the $(\eta\cdot t_{(8)})_1$ tensor of (5.4) and (5.8) is given in equation (B.2) and the expression for the $t_{(10)}$ tensor is given in appendix B.2 .

The expression for $A_{D^{2n}F^5}(1,2,3,4,5)$ in (5.2) is such that, besides containing no poles and being cyclic invariant, satisfies (on-shell) gauge invariance and world-sheet parity[15]

[15]The upper index (1) in all factors $H^{(1)}$, $P^{(1)}$, ... , denotes that they are invariant under a twisting

on *each* group of terms, that is, on $[H^{(1)} \cdot h^{(1)}(\zeta, k) + (\text{cyclic permutations})]$, ..., $[Z^{(1)} \cdot z^{(1)}(\zeta, k) + (\text{cyclic permutations})]$ and $\Delta \cdot \delta(\zeta, k)$, separately. These conditions are enough to find a local lagrangian for each of those terms, at each order in α'.

5.2 The main formula

From formula (5.2), for $A_{D^{2n}F^5}(1,2,3,4,5)$, and the kinematical expressions (5.3)–(5.8), the following effective lagrangian can be obtained for the $D^{2n}F^5$ terms, up to terms which are sensible to 6 or higher-point amplitudes:[16]

$$\begin{aligned}
\mathcal{L}_{D^{2n}F^5} = ig^3 & \int\!\!\int\!\!\int\!\!\int\!\!\int \left\{ \prod_{j=1}^{5} d^{10}x_j \; \delta^{(10)}(x - x_j) \right\} \times \\
& \times \Bigg[\Bigg\{ \frac{1}{32} H^{(1)}(-D_2 \cdot D_3, -D_3 \cdot D_4\,, -D_4 \cdot D_5)\; t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} + \\
& \quad + \frac{1}{16} P^{(1)}(-D_1 \cdot D_2, -D_2 \cdot D_3, -D_3 \cdot D_4\,, -D_4 \cdot D_5, -D_5 \cdot D_1) \times \\
& \quad \times (\eta \cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \Bigg\} \times \\
& \quad \times \operatorname{tr}\Big(F_{\mu_1\nu_1}(x_1) F_{\mu_2\nu_2}(x_2) F_{\mu_3\nu_3}(x_3) F_{\mu_4\nu_4}(x_4) F_{\mu_5\nu_5}(x_5) \Big) - \\
& \quad - U^{(1)}(-D_1 \cdot D_2, -D_2 \cdot D_3, -D_4 \cdot D_5, -D_5 \cdot D_1) \times \\
& \quad \times \Bigg\{ \frac{1}{64} t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \operatorname{tr}\Big(F_{\mu_1\nu_1}(x_1) F_{\mu_2\nu_2}(x_2) D^{\alpha} F_{\mu_3\nu_3}(x_3) D_{\alpha} F_{\mu_4\nu_4}(x_4) F_{\mu_5\nu_5}(x_5) \Big) + \\
& \qquad + \frac{1}{16} t_{(8)}^{\mu_4\nu_4\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2} \operatorname{tr}\Big(D^{\mu_3} F_{\mu_1\nu_1}(x_1) D^{\nu_3} F_{\mu_2\nu_2}(x_2) F_{\mu_3\nu_3}(x_3) F_{\mu_4\nu_4}(x_4) F_{\mu_5\nu_5}(x_5) \Big) + \\
& \qquad + \frac{1}{16} t_{(8)}^{\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5\mu_1\nu_1} \operatorname{tr}\Big(D^{\mu_2} F_{\mu_1\nu_1}(x_1) F_{\mu_2\nu_2}(x_2) D^{\nu_2} F_{\mu_3\nu_3}(x_3) F_{\mu_4\nu_4}(x_4) F_{\mu_5\nu_5}(x_5) \Big) - \\
& \qquad - \frac{1}{16} t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} \operatorname{tr}\Big(D^{\mu_5} F_{\mu_1\nu_1}(x_1) F_{\mu_2\nu_2}(x_2) F_{\mu_3\nu_3}(x_3) D^{\nu_5} F_{\mu_4\nu_4}(x_4) F_{\mu_5\nu_5}(x_5) \Big) - \\
& \qquad - \frac{1}{16} t_{(8)}^{\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \operatorname{tr}\Big(D^{\mu_4} F_{\mu_1\nu_1}(x_1) F_{\mu_2\nu_2}(x_2) F_{\mu_3\nu_3}(x_3) F_{\mu_4\nu_4}(x_4) D^{\nu_4} F_{\mu_5\nu_5}(x_5) \Big) \Bigg\} - \\
& \quad - \frac{1}{8} W^{(1)}(-D_1 \cdot D_2, -D_2 \cdot D_3, -D_3 \cdot D_4\,, -D_4 \cdot D_5, -D_5 \cdot D_1) \times \\
& \quad \times t_{(8)}^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \operatorname{tr}\Big(F_{\mu_1\nu_1}(x_1) D^{\mu_1} F_{\mu_2\nu_2}(x_2) F_{\mu_3\nu_3}(x_3) F_{\mu_4\nu_4}(x_4) D^{\nu_1} F_{\mu_5\nu_5}(x_5) \Big) - \\
& \quad - \frac{1}{8} Z^{(1)}(-D_1 \cdot D_2, -D_2 \cdot D_3, -D_3 \cdot D_4\,, -D_4 \cdot D_5, -D_5 \cdot D_1) \times \\
& \quad \times t_{(8)}^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \Bigg\{ \operatorname{tr}\Big(F_{\mu_1\nu_1}(x_1) F_{\mu_2\nu_2}(x_2) D^{\mu_1} F_{\mu_3\nu_3}(x_3) D^{\nu_1} F_{\mu_4\nu_4}(x_4) F_{\mu_5\nu_5}(x_5) \Big) - \\
& \qquad\qquad - \operatorname{tr}\Big(F_{\mu_1\nu_1}(x_1) D^{\mu_1} F_{\mu_2\nu_2}(x_2) F_{\mu_3\nu_3}(x_3) F_{\mu_4\nu_4}(x_4) D^{\nu_1} F_{\mu_5\nu_5}(x_5) \Big) \Bigg\} + \\
& \quad + \frac{1}{160} \Delta(-D_1 \cdot D_2, -D_2 \cdot D_3, -D_3 \cdot D_4\,, -D_4 \cdot D_5, -D_5 \cdot D_1) \times \\
& \quad \times \Bigg\{ t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \times \\
& \qquad \times \operatorname{tr}\Big(D^{\alpha} D^{\beta} F_{\mu_1\nu_1}(x_1) D_{\alpha} F_{\mu_2\nu_2}(x_2) F_{\mu_3\nu_3}(x_3) F_{\mu_4\nu_4}(x_4) D_{\beta} F_{\mu_5\nu_5}(x_5) \Big) + \\
& \qquad + 4(\eta \cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \times \\
& \qquad \times \operatorname{tr}\Big(F_{\mu_1\nu_1}(x_1) D_{\alpha} F_{\mu_2\nu_2}(x_2) D^{\alpha} F_{\mu_3\nu_3}(x_3) D_{\beta} F_{\mu_4\nu_4}(x_4) D^{\beta} F_{\mu_5\nu_5}(x_5) \Big) \Bigg\} \Bigg] .
\end{aligned} \tag{5.9}$$

transformation with respect to index 1. Similarly, the upper index (1) in all kinematical expressions $h^{(1)}(\zeta, k)$, $p^{(1)}(\zeta, k)$, ..., denotes that they change their sign under a twisting transformation with respect to index 1.

[16]When considering the α' expansion of eq. (5.9), whenever two covariant derivatives D_α and D_β (with $\alpha \neq \beta$) operate on a *same* field strength $F_{\mu\nu}(x)$, the order in which they operate does not matter since the difference between the two possibilities will be sensible only to 6 or higher-point amplitudes.

Notice that in this result we have used that the factors $P^{(1)}$, $W^{(1)}$, $Z^{(1)}$ and Δ depend in *all* five (independent) α_{ij} variables, while $H^{(1)}$ and $U^{(1)}$ depend on a *lower* number of them:

$$\begin{aligned} H^{(1)} &= H^{(1)}(\alpha_{23}, \alpha_{34}, \alpha_{45})\,, && (5.10)\\ U^{(1)} &= U^{(1)}(\alpha_{12}, \alpha_{23}, \alpha_{45}, \alpha_{51})\,, && (5.11)\end{aligned}$$

as may be seen directly in the formulas of appendix C.3.

That formula (5.9) is indeed correct may be confirmed by calculating the 1 PI 5-point function of it, $\Gamma^{(5)\ a_1 a_2 a_3 a_4 a_5}_{\mu_1\mu_2\mu_3\mu_4\mu_5}(x_1, x_2, x_3, x_4, x_5)$, and then calculating its Fourier Transform[17]. From this last one it may be checked that the corresponding subamplitude is the one given in (5.2).

5.3 Some α' terms

In this subsection we will see two applications of formula (5.9), in order to see how it works. Before doing so, notice that the α' dependent factors of (5.2) and (5.9) do not begin their power expansions at the same order in α'. As may be seen from the explicit expansions of them in appendix C.3: $H^{(1)}$ and $P^{(1)}$ begin at $\mathcal{O}(\alpha'^3)$ order; $U^{(1)}$, $W^{(1)}$ and $Z^{(1)}$ begin at $\mathcal{O}(\alpha'^4)$ order and Δ begins at $\mathcal{O}(\alpha'^5)$ order.

As a first application in which the 5-point amplitude is important, let us see the case of the non-abelian effective lagrangian of the open superstring at order $\mathcal{O}(\alpha'^3)$ (which was first completely and correctly calculated in [29]). At this order, as may be seen in (2.4) and (2.5), the lagrangian contains F^5 and D^2F^4 terms. The first ones can be taken from the lagrangian in (5.9) and the second ones from the lagrangian in (3.7), giving:

$$\begin{aligned} {\mathcal{L}_{\text{ef}}}^{(3)} = \zeta(3)\ \alpha'^3 \Bigg[&-i\frac{g^3}{4}\Big\{ t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} + 2(\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} \Big\} \times \\ &\times \text{tr}\Big(F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}F_{\mu_5\nu_5} \Big) \ + \frac{g^2}{2} t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} \times \\ &\times \Big\{ \text{tr}\Big(D^\alpha F_{\mu_1\nu_1} D_\alpha F_{\mu_2\nu_2} F_{\mu_3\nu_3} F_{\mu_4\nu_4} \Big) + \\ &\quad + \text{tr}\Big(D^2 F_{\mu_1\nu_1} F_{\mu_2\nu_2} F_{\mu_3\nu_3} F_{\mu_4\nu_4} \Big) \Big\} \Bigg]\,. \end{aligned} \tag{5.12}$$

In order to compare (5.12) with one of the expressions of the known result we use that

$$D^2F_{\mu\nu} = D^\alpha(D_\alpha F_{\mu\nu}) = D_\mu(D_\alpha F^\alpha{}_\nu) - D_\nu(D_\alpha F^\alpha{}_\mu) + 2\ i\ g\ [F_{\alpha\mu}, F_\nu{}^\alpha]\,, \tag{5.13}$$

which may be derived using the Bianchi identity and the $[D, D]F = -ig[F, F]$ relation (see (A.8) and (A.7), respectively).

[17]In the case of the 1 PI 4-point function and the corresponding subamplitude, for example, this was done in the appendices of [17] and [35].

JHEP03(2005)055

Substituting (5.13) in (5.12), and dropping out on-shell terms (i.e. the ones which contain $D_\alpha F^\alpha{}_\mu$)[18] leads to

$$\mathcal{L}_{\rm ef}{}^{(3)} = \zeta(3)\;\alpha'^3\Bigg[\,-i\frac{g^3}{4}(t_{(10)}+4(\eta\cdot t_{(8)})_1)^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\times$$
$$\times\,\mathrm{tr}\Big(F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)\;+$$
$$+\;\frac{g^2}{2}t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\,\mathrm{tr}\Big(D^\alpha F_{\mu_1\nu_1}D_\alpha F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}\Big)\Bigg]\,. \tag{5.14}$$

Using the explicit expressions of the $t_{(8)}$ and the $t_{(10)}$ tensors, and a basis of F^5 and D^2F^4 terms taken from [40][19], it may be verified that the D^2F^4 terms of (5.14) agree with the ones in [27, eq. (4.17)] and that the F^5 terms of (5.14) agree, on-shell, with the ones of the mentioned equation after going to $D=4$.[20] In this last reference it was seen that the lagrangian in equation (4.17) of it agrees (in $D=4$) with the one in [29], which was also confirmed in [30] and [34].

The next interesting example consists in the $\mathcal{O}(\alpha'^4)$ terms of the effective lagrangian which are sensible to 5-point amplitudes. We take the D^2F^5 terms from (5.9) and the D^4F^4 ones from the lagrangian in (3.7), giving:

$$\mathcal{L}'^{(4)} = \mathcal{L}_{\rm D^2F^5} + \mathcal{L}_{\rm D^4F^4}\,, \tag{5.15}$$

where

$$\mathcal{L}_{\rm D^2F^5} =$$
$$= i\frac{\pi^4}{90}g^3\alpha'^4\Bigg[-\frac{1}{8}t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\times$$
$$\times\Big\{4\,\mathrm{tr}\Big(F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}D^\alpha F_{\mu_4\nu_4}D_\alpha F_{\mu_5\nu_5}\Big)+$$
$$+\,4\,\mathrm{tr}\Big(F_{\mu_1\nu_1}D^\alpha F_{\mu_2\nu_2}D_\alpha F_{\mu_3\nu_3}F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)+$$
$$+\,\mathrm{tr}\Big(F_{\mu_1\nu_1}F_{\mu_2\nu_2}D^\alpha F_{\mu_3\nu_3}D_\alpha F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)\Big\}-$$
$$-\,\frac{1}{4}(\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\times$$
$$\times\Big\{4\,\mathrm{tr}\Big(F_{\mu_1\nu_1}F_{\mu_2\nu_2}D^\alpha F_{\mu_3\nu_3}D_\alpha F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)+$$
$$+\,\mathrm{tr}\Big(F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}D^\alpha F_{\mu_4\nu_4}D_\alpha F_{\mu_5\nu_5}\Big)+$$

[18] In [3] it was seen, at least up to $\mathcal{O}(\alpha'^2)$ order, that the on-shell terms in the effective lagrangian are not sensible to on-shell scattering amplitudes.

[19] See appendix B of this reference.

[20] In $D=4$, due to the Cayley-Hamilton theorem, there is an identity involving the F^5 terms which reduces the number of them which are independent. That is the reason of why our F^5 terms in (5.14) only agree with the ones which go in the effective action of $N=4$ SYM in [27] after going to 4 dimensions. We thank D. Grasso for explaining this to us.

JHEP03(2005)055

$$
\begin{aligned}
&+\operatorname{tr}\Big(F_{\mu_1\nu_1}D^\alpha F_{\mu_2\nu_2}D_\alpha F_{\mu_3\nu_3}F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)+\\
&+4\operatorname{tr}\Big(D^\alpha F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}D_\alpha F_{\mu_5\nu_5}\Big)+\\
&+4\operatorname{tr}\Big(D^\alpha F_{\mu_1\nu_1}D_\alpha F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)\Big\}-\\
&-\frac{1}{4}t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\operatorname{tr}\Big(F_{\mu_1\nu_1}F_{\mu_2\nu_2}D^\alpha F_{\mu_3\nu_3}D_\alpha F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)-\\
&-t_{(8)}^{\mu_4\nu_4\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2}\operatorname{tr}\Big(D^{\mu_3}F_{\mu_1\nu_1}D^{\nu_3}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)-\\
&-t_{(8)}^{\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5\mu_1\nu_1}\operatorname{tr}\Big(D^{\mu_2}F_{\mu_1\nu_1}F_{\mu_2\nu_2}D^{\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)+\\
&+t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\operatorname{tr}\Big(D^{\mu_5}F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}D^{\nu_5}F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)+\\
&+t_{(8)}^{\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\operatorname{tr}\Big(D^{\mu_4}F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}D^{\nu_4}F_{\mu_5\nu_5}\Big)-\\
&-\frac{1}{2}t_{(8)}^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\operatorname{tr}\Big(F_{\mu_1\nu_1}D^{\mu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}D^{\nu_1}F_{\mu_5\nu_5}\Big)+\\
&+\frac{1}{8}t_{(8)}^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\Big\{\operatorname{tr}\Big(F_{\mu_1\nu_1}F_{\mu_2\nu_2}D^{\mu_1}F_{\mu_3\nu_3}D^{\nu_1}F_{\mu_4\nu_4}F_{\mu_5\nu_5}\Big)-\\
&\qquad-\operatorname{tr}\Big(F_{\mu_1\nu_1}D^{\mu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}D^{\nu_1}F_{\mu_5\nu_5}\Big)\Big\}\Big] \qquad (5.16)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{\mathrm{D^4F^4}}=&\frac{\pi^4}{2880}\,g^2\,\alpha'^4\,t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\times\\
&\times\Big[9\operatorname{tr}\Big(D^2D^2F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}\Big)+\\
&+16\operatorname{tr}\Big(D^\alpha F_{\mu_1\nu_1}D_\alpha F_{\mu_2\nu_2}D^\beta F_{\mu_3\nu_3}D_\beta F_{\mu_4\nu_4}\Big)+\\
&+4\operatorname{tr}\Big(D^\alpha F_{\mu_1\nu_1}D_{(\beta}D_{\alpha)}F_{\mu_2\nu_2}D^\beta F_{\mu_3\nu_3}F_{\mu_4\nu_4}\Big)+\\
&+18\operatorname{tr}\Big(D^2F_{\mu_1\nu_1}D^2F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}\Big)+9\operatorname{tr}\Big(D^2F_{\mu_1\nu_1}F_{\mu_2\nu_2}D^2F_{\mu_3\nu_3}F_{\mu_4\nu_4}\Big)+\\
&+16\operatorname{tr}\Big(D^\alpha D^\beta F_{\mu_1\nu_1}D_\alpha D_\beta F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}\Big)+\\
&+34\operatorname{tr}\Big(D^{(2}D^{\beta)}F_{\mu_1\nu_1}D_\beta F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}\Big)+\\
&+34\operatorname{tr}\Big(D^2F_{\mu_1\nu_1}F_{\mu_2\nu_2}D^\beta F_{\mu_3\nu_3}D_\beta F_{\mu_4\nu_4}\Big)\Big]. \qquad (5.17)
\end{aligned}
$$

At this order the symmetrized prescription mentioned in (3.9) begins to apply: that is why some terms in $\mathcal{L}_{\mathrm{D^4F^4}}$ contain symmetrized products. Using the explicit expressions of the $t_{(8)}$ and the $t_{(10)}$ tensors, integration by parts, the Bianchi and the $[D,D]F=-ig[F,F]$

identities, and dropping out the on-shell terms, the expressions in (5.16) and (5.17) could be made explicit and reduced to a minimum length, but we will leave that work to be done somewhere else and just content ourselves with those expressions as they are, in order to see how formulas (5.9) and (3.7) have operated at this order in α'. In any case, we have checked that the abelian limit of (5.17) agrees completely with the $\partial^4 F^4$ terms of [41].

The lagrangian in (5.15) (with the expressions in (5.16) and (5.17)) is not the complete lagrangian at $\mathcal{O}(\alpha'^4)$ because the F^6 terms should also be present (that is why we have called it $\mathcal{L}'^{(4)}$ instead of $\mathcal{L}^{(4)}$). A complete lagrangian at this order has been determined in [19] and confirmed in [20] and [21].

The procedure to continue writing the $D^{2n}F^5$ terms from (5.9) and the $D^{2n}F^4$ ones from (3.7) is quite direct using the expansions of all the α' dependent factors. In the case of the Gamma factor, the coefficients of the α' expansion are known up to infinity, while in this work we have expanded the T and $(2\alpha')^2K_3$ factors (from which the $P^{(1)}$, $U^{(1)}$, $W^{(1)}$ and Δ factors have been obtained) only up to $\mathcal{O}(\alpha'^6)$ order. So in this section we could, in principle, go on with the $D^{2n}F^5$ terms up to that order, but to save space we will not do it here.

In spite of what was mentioned in the previous paragraph, the structure of the α' expansions of T and $(2\alpha')^2K_3$ is *completely* known since, as it is proved in detail in appendix C.2, at each order in α' only cyclic polynomial invariants (in the α_{ij} variables) appear.[21] So the only unknowns are the coefficients which go with each of those cyclic polynomial invariants: this is a considerably much reduced number of unknowns than the ones which in principle should be determined at every α' order (see appendix C.2.5 for further details).

6. Summary and final remarks

We have succeeded in finding the $D^{2n}F^5$ terms of the open superstring effective lagrangian, to all order in α'. This lagrangian is of importance in the corresponding sector of the SO(32) Type I theory and in the description of the low energy interaction of D-branes. To our knowledge, this is the second *non-abelian* result, reported in the literature, which is complete (in the sense that all the terms in the lagrangian and its coefficients are given in a closed form) and which has been determined to all order in α'; the first one being the determination of the $D^{2n}F^4$ in [35].

This has been a very much more complicated problem than the determination of the $D^{2n}F^4$ terms, which consisted in a non-abelian generalization of a previous result in [17]. We have found the general formula and we have given expansions which allow us to compute explicitly the $D^{2n}F^5$ terms up to $\mathcal{O}(\alpha'^6)$ order. We have checked that the F^5 terms of our lagrangian, together with the D^2F^4 ones found previously (see eq. (3.7)), agree with the known result at $\mathcal{O}(\alpha'^3)$ order. At $\mathcal{O}(\alpha'^4)$ order, it would be interesting to see if our D^2F^5 and D^4F^4 terms agree with the ones which are sensible to 4 and 5-point amplitudes in [20, eqs. (1.5) and (1.6)].

[21] Further more, the polynomials which appear at each order in α' should also be invariant under a twisting transformation, so this diminishes a little the number of them which appear in the expansion. In appendix C.2.5 we give a complete list of them up to sixth degree [42].

JHEP03(2005)055

The starting point of all this huge labor has been the determination of the 5-point subamplitude of massless bosons in Open Superstring Theory, which we have found in terms of two kinematical expressions and two α' dependent factors (see eq. (4.13)).

Another important step that we have done, in order to be able to find all the $D^{2n}F^5$ terms, has been the determination of the scattering subamplitude $A_{D^{2n}F^5}(1,2,3,4,5)$ as a sum of terms which have no poles and have manifest cyclic and (on-shell) gauge invariance, as well as world-sheet parity symmetry (see eq. (5.2)). We have done this by using the known $t_{(8)}$ and a new $t_{(10)}$ tensor in all the kinematical expressions involved in that subamplitude.

The explicit expression of $\mathcal{L}_{D^{2n}F^5}$ depends directly on the previously found expression of $\mathcal{L}_{D^{2n}F^4}$, because the scattering subamplitude $A_{D^{2n}F^5}(1,2,3,4,5)$ is calculated in terms of $A_{D^{2n}F^4}(1,2,3,4,5)$ (see eq. (5.1)). Now, given that there is not a unique way in choosing the terms of $\mathcal{L}_{D^{2n}F^4}$,[22] and the fact that the α' factors in (5.2) are not all independent (see, for example, eq. (F.12)), it may happen that a final lagrangian $\mathcal{L}_{D^{2n}F^5}$ could eventually be found in terms of only the $t_{(10)}$ tensor, as mentioned in [36]. In the present work we have not look further in this direction.

The calculations of this work have been extremely long and, based on the experience we have had in solving them out, we think it is impossible to have done them without any computer assistance. In spite of this fact, and contrary to what it is generally believed, the main result of our paper suggests that it is indeed possible to take to the level of the effective lagrangian, to all order in α', the information of (tree level) superstring scattering amplitudes: at least we have succeeded on this subject, up to 5-field terms, in the case of Open Superstring Theory.

On a future paper [43] we will use the results of the present work to determine the tree level $\alpha'^4 R^5$ terms in the effective lagrangians of the type II theories, by means of the KLT relations [44], and we will consider the possibility of finding and *all* α' result for those actions, as it was done in [35] in the case of the 4-Riemann tensor terms.

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A. Conventions and identities

1. Metric, symmetrization and antisymmetrization over spacetime indexes.

   We use the following convention for the Minkowski metric:

$$\eta_{\mu\nu} = \text{diag}(-,+,\ldots,+)\,. \tag{A.1}$$

[22] Since the only requirement for this lagrangian is to reproduce superstring 4-point amplitudes, there is some freedom in the way its covariant derivatives are present on it.

The symmetrization and antisymmetrization convention that we use, on the space-time indexes of a product of two vectors A and B, is the following:

$$A^{(\mu}B^{\nu)} = \frac{1}{2}(A^\mu B^\nu + A^\nu B^\mu)\,, \tag{A.2}$$

$$A^{[\mu}B^{\nu]} = \frac{1}{2}(A^\mu B^\nu - A^\nu B^\mu)\,. \tag{A.3}$$

2. Gauge group generators, field strength and covariant derivative.

   Gauge fields are matrices in the Lie group internal space, so that $A_\mu = A^\mu{}_a \lambda^a$, where the λ^a are the generators (in a matrix representation) which satisfy the usual relation

$$\mathrm{tr}(\lambda^a \lambda^b) = \delta^{ab}\,. \tag{A.4}$$

   The field strength and the covariant derivative are defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\,, \tag{A.5}$$

$$D_\mu \phi = \partial_\mu \phi - ig[A_\mu, \phi]\,, \tag{A.6}$$

   and they are related by the identity

$$[D_\mu, D_\nu]\phi = -ig\ [F_{\mu\nu}, \phi]\,. \tag{A.7}$$

   Covariant derivatives of field strengths satisfy the Bianchi identity:

$$D^\mu F^{\nu\rho} + D^\rho F^{\mu\nu} + D^\nu F^{\rho\mu} = 0\,. \tag{A.8}$$

B. Tensors

B.1 $t_{(8)}$ and $(\eta \cdot t_{(8)})_1$ tensors

The $t_{(8)}$ tensor[23], characteristic of the 4 boson scattering amplitude, is antisymmetric on each pair (μ_j, ν_j) $(j = 1, 2, 3, 4)$ and is symmetric under any exchange of such of pairs. It satisfies the identity:[24]

$$\begin{aligned} & t^{(8)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} A_1^{\mu_1\nu_1} A_2^{\mu_2\nu_2} A_3^{\mu_3\nu_3} A_4^{\mu_4\nu_4} = \\ & \quad = -2\Big(\mathrm{Tr}(A_1A_2)\,\mathrm{Tr}(A_3A_4) + \mathrm{Tr}(A_1A_3)\,\mathrm{Tr}(A_2A_4) + \mathrm{Tr}(A_1A_4)\,\mathrm{Tr}(A_2A_3)\Big) + \\ & \qquad + 8\Big(\mathrm{Tr}(A_1A_2A_3A_4) + \mathrm{Tr}(A_1A_3A_2A_4) + \mathrm{Tr}(A_1A_3A_4A_2)\Big)\,, \end{aligned} \tag{B.1}$$

where the A_j tensors are antisymmetric and where 'Tr' means the trace over the spacetime indexes.

[23] An explicit expression for it may be found in [37, equation (4.A.21)].
[24] Formula (B.1) has been taken from [17, appendix A].

A ten index tensor, which is also antisymmetric on each pair (μ_j,ν_j), can be constructed from the Minkowski metric tensor and the $t_{(8)}$ one, as follows:

$$(\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} = \eta^{\nu_3\nu_4}t_{(8)}^{\mu_3\mu_4\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2} + \eta^{\mu_3\mu_4}t_{(8)}^{\nu_3\nu_4\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2} - \\ - \eta^{\mu_3\nu_4}t_{(8)}^{\nu_3\mu_4\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2} - \eta^{\nu_3\mu_4}t_{(8)}^{\mu_3\nu_4\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2}\,. \quad \text{(B.2)}$$

This tensor appears in the 5-point amplitude of the open superstring. It also changes sign under a twisting transformation[25] with respect to index 1, that is,

$$(\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_5\nu_5\mu_4\nu_4\mu_3\nu_3\mu_2\nu_2} = -(\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\,. \quad \text{(B.3)}$$

We have used the subindex 1 in the $(\eta\cdot t_{(8)})$ tensor as a reminder of this relation.

B.2 $t_{(10)}$ tensor

The $t_{(10)}$ tensor is another ten index tensor that appears in the 5-point amplitude of the open superstring. It is linearly independent to the $(\eta\cdot t_{(8)})_1$ one. It can be constructed by the following procedure:

1. Using the Minkowski metric tensor, we make all possible ten index tensorial structures:

   $\eta^{\mu_i\nu_n}\eta^{\mu_j\nu_o}\eta^{\mu_k\nu_p}\eta^{\mu_l\nu_q}\eta^{\mu_m\nu_r}$, $\eta^{\mu_i\nu_n}\eta^{\mu_j\nu_o}\eta^{\mu_k\nu_p}\eta^{\mu_l\mu_m}\eta^{\nu_r\nu_q}$ and $\eta^{\mu_i\nu_n}\eta^{\mu_j\mu_k}\eta^{\nu_p\nu_o}\eta^{\mu_l\mu_m}\eta^{\nu_r\nu_q}$,

   with $i\neq j\neq k\neq l\neq m$ and $n\neq o\neq p\neq q\neq r$. These indexes are in the range $1,\ldots,5$. When applied to the general expression $\zeta^1_{\mu_1}k^1_{\nu_1}\zeta^2_{\mu_2}k^2_{\nu_2}\zeta^3_{\mu_3}k^3_{\nu_3}\zeta^4_{\mu_4}k^4_{\nu_4}\zeta^5_{\mu_5}k^5_{\nu_5}$, many terms will be null due to the physical state condition ($\zeta^i\cdot k^i=0$). Despising these terms, 544 structures remain.

2. Making a linear combination of these 544 terms, we demand antisymmetry on each pair (μ_j,ν_j) and also a change of sign under a twisting transformation with respect to index 1 (the same as in eq. (B.3)). This procedure results in a ten index tensor with only 14 free parameters, which will be our *ansatz* for the $t_{(10)}$ tensor.

   Unlike the $t_{(8)}$ tensor, the $t_{(10)}$ one *is not cyclic invariant* in the pair of indexes (μ_j,ν_j).[26]

3. Then, the last 14 parameters are determined by substituting $t_{(10)}$ in the kinematical expression $h^{(1)}(\zeta,k)$, in (5.3), and comparing it with the corresponding kinematical structure in eq. (F.15) of appendix F. This comparison is not immediate since both expressions only agree after using on-shell and physical state conditions, together with momentum conservation. At the end we obtain an expression which satisfies an

[25] See the third item of subsection 4.2 for further details about a twisting transformation on the disk.

[26] Due to the symmetries of the 4-point subamplitude, the $t_{(8)}$ tensor is expected to be cyclic invariant in the pair of indexes (μ_j,ν_j), but in fact it happens to be *completely* symmetric with respect to those indexes[37].

identity similar to that in (B.1):

$$
\begin{aligned}
& t^{(10)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5} A_1^{\mu_1\nu_1} A_2^{\mu_2\nu_2} A_3^{\mu_3\nu_3} A_4^{\mu_4\nu_4} A_5^{\mu_5\nu_5} = \\
&\quad = -8\Big[\mathrm{Tr}(A_1A_2)\,\mathrm{Tr}(A_3A_4A_5) + \mathrm{Tr}(A_1A_3)\,\mathrm{Tr}(A_2A_4A_5) + \mathrm{Tr}(A_1A_4)\,\mathrm{Tr}(A_2A_3A_5) + \\
&\qquad + \mathrm{Tr}(A_1A_5)\,\mathrm{Tr}(A_2A_3A_4) + \mathrm{Tr}(A_2A_3)\,\mathrm{Tr}(A_1A_4A_5) + \mathrm{Tr}(A_2A_4)\,\mathrm{Tr}(A_1A_3A_5) + \\
&\qquad + \mathrm{Tr}(A_2A_5)\,\mathrm{Tr}(A_1A_3A_4) + \mathrm{Tr}(A_3A_4)\,\mathrm{Tr}(A_1A_2A_5) + \mathrm{Tr}(A_3A_5)\,\mathrm{Tr}(A_1A_2A_4) + \\
&\qquad + \mathrm{Tr}(A_4A_5)\,\mathrm{Tr}(A_1A_2A_3)\Big] + 48\ \mathrm{Tr}(A_1A_2A_3A_4A_5)\ + \\
&\quad + 16\Big[\mathrm{Tr}(A_1A_2A_3A_5A_4) + \mathrm{Tr}(A_1A_2A_4A_3A_5) + \mathrm{Tr}(A_1A_2A_5A_3A_4) + \\
&\qquad + \mathrm{Tr}(A_1A_2A_4A_5A_3) - \mathrm{Tr}(A_1A_2A_5A_4A_3) + \mathrm{Tr}(A_1A_3A_2A_4A_5) - \\
&\qquad - \mathrm{Tr}(A_1A_3A_2A_5A_4) + \mathrm{Tr}(A_1A_4A_2A_3A_5) + \mathrm{Tr}(A_1A_5A_2A_3A_4) - \\
&\qquad - \mathrm{Tr}(A_1A_4A_2A_5A_3) - \mathrm{Tr}(A_1A_5A_2A_4A_3)\Big],
\end{aligned} \tag{B.4}
$$

where the A_j fields are antisymmetric. From (B.4) an explicit expression of the $t_{(10)}$ tensor may be obtained, once its symmetry properties are considered.

C. α' dependent factors

C.1 Gamma factor

As remarked in [30], using the Taylor expansion for $\ln\ \Gamma(1+z)$,[27]

$$
\ln\Gamma(1+z) = -\gamma z + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} z^k \qquad (-1 < z \leq 1)\,, \tag{C.1}
$$

it may be proved that the explicit α' expansion for the Gamma factor in eq. (3.2) is given by

$$
\alpha'^2 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} = \frac{1}{st}\cdot \exp\left\{ \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} \alpha'^k (s^k + t^k - (s+t)^k) \right\}. \tag{C.2}
$$

Up to $\mathcal{O}(\alpha'^6)$ terms this gives:

$$
\begin{aligned}
\alpha'^2 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} &= \frac{1}{st} - \frac{\pi^2}{6}\alpha'^2 - \zeta(3)(s+t)\ \alpha'^3 - \frac{\pi^4}{360}(4s^2+st+4t^2)\ \alpha'^4 + \\
&+ \left[\frac{\pi^2}{6}\zeta(3)\ st(s+t) - \zeta(5)(s^3+2s^2t+2st^2+t^3) \right] \alpha'^5 + \\
&+ \left[\frac{1}{2}\zeta(3)^2 st(s+t)^2 - \right. \\
&\qquad \left. - \frac{\pi^6}{15120}(16s^4+12s^3t+23s^2t^2+12st^3+16t^4) \right] \alpha'^6 \\
&+ \mathcal{O}(\alpha'^7)\,.
\end{aligned} \tag{C.3}
$$

[27] See formula (10.44c) of [45], for example.

The function $f(s,t)$, defined in (3.8), consists in this Gamma factor (divided by α'^2) with the pole subtracted, so it has a well defined power series:

$$f(s,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\ s^m t^n\ \alpha'^{m+n}\,, \tag{C.4}$$

where $a_{mn} = a_{nm}$, with $a_{00} = -\pi^2/6$, $a_{01} = -\zeta(3)$, $a_{11} = -\pi^4/360$, $a_{02} = -\pi^4/90$, etc.

C.2 K_2, K_3 and T factors

In this section we briefly review the definitions of the factors K_2 and K_3, given in [34, appendix A.1] and we deal with the factor T introduced in eq. (4.10) of this work. We study the twisting and the cyclic symmetry and see how they are present in the α' expansion of K_3 and T.

C.2.1 The definitions and the α_{ij} variables

The factors K_2 and K_3 were defined in [34] as

$$K_2 = \int_0^1 dx_3 \int_0^{x_3} dx_2 x_3^{2\alpha'\alpha_{13}}(1-x_3)^{2\alpha'\alpha_{34}-1}x_2^{2\alpha'\alpha_{12}-1}(1-x_2)^{2\alpha'\alpha_{24}}(x_3-x_2)^{2\alpha'\alpha_{23}}\,, \tag{C.5}$$

$$K_3 = \int_0^1 dx_3 \int_0^{x_3} dx_2 x_3^{2\alpha'\alpha_{13}-1}(1-x_3)^{2\alpha'\alpha_{34}}x_2^{2\alpha'\alpha_{12}}(1-x_2)^{2\alpha'\alpha_{24}-1}(x_3-x_2)^{2\alpha'\alpha_{23}}\,. \tag{C.6}$$

After making the substitution $x_2 = u\cdot x_3$ in the inner integral (keeping x_3 constant) they become

$$K_2 = \int_0^1 dx_3 \int_0^1 du\ x_3^{2\alpha'\rho}(1-x_3)^{2\alpha'\alpha_{34}-1}u^{2\alpha'\alpha_{12}-1}(1-ux_3)^{2\alpha'\alpha_{24}}(1-u)^{2\alpha'\alpha_{23}}\,, \tag{C.7}$$

$$K_3 = \int_0^1 dx_3 \int_0^1 du\ x_3^{2\alpha'\rho}(1-x_3)^{2\alpha'\alpha_{34}}u^{2\alpha'\alpha_{12}}(1-ux_3)^{2\alpha'\alpha_{24}-1}(1-u)^{2\alpha'\alpha_{23}}\,. \tag{C.8}$$

Here,

$$\rho = \alpha_{12} + \alpha_{13} + \alpha_{23}\,. \tag{C.9}$$

As was seen in [8], these factors may be written in terms of Euler Beta functions and a generalized Hypergeometric function, as

$$\begin{aligned} K_2 &= B(2\alpha'\alpha_{12}, 1+2\alpha'\alpha_{23})\cdot B(2\alpha'\alpha_{34}, 1+2\alpha'\rho)\,\cdot \\ &\quad \cdot\,{}_3F_2(1+2\alpha'\rho, 2\alpha'\alpha_{12}, -2\alpha'\alpha_{24};\ 1+2\alpha'\rho+2\alpha'\alpha_{34}, 1+2\alpha'\alpha_{12}+2\alpha'\alpha_{23};\ 1)\,, \end{aligned} \tag{C.10}$$

$$\begin{aligned} K_3 &= B(1+2\alpha'\alpha_{12}, 1+2\alpha'\alpha_{23})\cdot B(1+2\alpha'\alpha_{34}, 1+2\alpha'\rho)\,\cdot \\ &\quad \cdot\,{}_3F_2(1+2\alpha'\rho, 1+2\alpha'\alpha_{12}, 1-2\alpha'\alpha_{24};\ 2+2\alpha'\rho+2\alpha'\alpha_{34}, 2+2\alpha'\alpha_{12}+2\alpha'\alpha_{23};\ 1)\,. \end{aligned} \tag{C.11}$$

We will not use (C.10) and (C.11) to find any α' expansion in the present work. In [34], using (C.7) and (C.8), the α' expansions of K_2 and K_3 were found up to $\mathcal{O}(\alpha')$ terms. The double integrals that appear at each power of α' were calculated using Harmonic Polylogarithms [46].

It is important to note that the ten α_{ij} variables defined in (4.5) are not all independent once the on-shell (${k_i}^2 = 0$) and the momentum conservation conditions are taken into account. In fact, only five of them are independent. When finding the α' expansions of the factors K_2 and K_3 we have chosen α_{12}, α_{23}, α_{34}, α_{45} and α_{51}, as independent variables. The rest of them are given by

$$\alpha_{13} = \alpha_{45} - \alpha_{12} - \alpha_{23} \tag{C.12}$$
$$\alpha_{14} = \alpha_{23} - \alpha_{51} - \alpha_{45} \tag{C.13}$$
$$\alpha_{24} = \alpha_{51} - \alpha_{23} - \alpha_{34} \tag{C.14}$$
$$\alpha_{25} = \alpha_{34} - \alpha_{12} - \alpha_{51} \tag{C.15}$$
$$\alpha_{35} = \alpha_{12} - \alpha_{45} - \alpha_{34}\,. \tag{C.16}$$

Notice that considering (C.12), the ρ variable in (C.9) coincides with α_{45}.

The T factor has already been defined in eq. (4.10) as

$$T = (2\alpha')^2 \Big[\alpha_{12}\alpha_{34}\ K_2 + (\alpha_{51}\ \alpha_{12} - \alpha_{12}\alpha_{34} + \alpha_{34}\ \alpha_{45}) K_3 \Big]\,.$$

T and K_3 are related to the Gamma factor of the 4-point amplitude by

$$(2\alpha')^2 \frac{\Gamma(2\alpha'\alpha_{34})\ \Gamma(2\alpha'\alpha_{45})}{\Gamma(1 + 2\alpha'\alpha_{34} + 2\alpha'\alpha_{45})} = \frac{1}{\alpha_{34}\alpha_{45}} T\Big|_{\alpha_{12}=0} - (2\alpha')^2 K_3\Big|_{\alpha_{12}=0}\,. \tag{C.17}$$

To see this we notice that using the definition of T, already mentioned, the relation in (C.17) is equivalent to

$$\{\alpha_{12}\ K_2\}\Big|_{\alpha_{12}=0} = \alpha_{45} \frac{\Gamma(2\alpha'\alpha_{34})\Gamma(2\alpha'\alpha_{45})}{\Gamma(1 + 2\alpha'\alpha_{34} + 2\alpha'\alpha_{45})}\,, \tag{C.18}$$

which can be proved using (C.10) and the fact that the Hypergeometric function appearing there becomes 1 when $\alpha_{12} = 0$.

The following relations (and the analog ones, obtained by cyclic permutations of them) can also be proved:

$$(2\alpha')^2 K_3\Big|_{\alpha_{12}=\alpha_{23}=0} = -(2\alpha')^2 \frac{\alpha_{34} f(-2\alpha_{34}, -2\alpha_{45}) - \alpha_{51} f(-2\alpha_{45}, -2\alpha_{51})}{\alpha_{34} - \alpha_{51}}\,, \tag{C.19}$$
$$T\Big|_{\alpha_{12}=\alpha_{23}=0} - 1 = -(2\alpha')^2 \alpha_{34}\alpha_{45}\alpha_{51} \frac{f(-2\alpha_{34}, -2\alpha_{45}) - f(-2\alpha_{45}, -2\alpha_{51})}{\alpha_{34} - \alpha_{51}}\,, \tag{C.20}$$
$$T\Big|_{\alpha_{12}=\alpha_{34}=0} - 1 = 0\,. \tag{C.21}$$

C.2.2 Twisting symmetry

The twisting transformation was already commented (in the third item of subsection 4.2) to be equivalent to a parity transformation in the string world-sheet. In this section we prove that K_2, K_3 and T are invariant under a twisting transformation with respect to the fifth insertion on the disk, that is, they are invariant under the transformation of indexes given in (4.15). For this purpose we use that

$$\int_0^1 dx_3 \int_0^{x_3} dx_2 \; g(x_2, x_3) = \int_0^1 dx_3 \int_0^{x_3} dx_2 \; g(1-x_3, 1-x_2)\,. \tag{C.22}$$

This can be easily proved by first changing the order of integration on the left integral and then making the substitution

$$\begin{cases} x_2 \;=\; 1 - x_3{}' \\ x_3 \;=\; 1 - x_2{}' \end{cases}. \tag{C.23}$$

Applying the result in (C.22) to the integrals in (C.5) and (C.6) we have the desired relation

$$K_i(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34}) = K_i(\alpha_{34};\alpha_{24};\alpha_{23};\alpha_{13};\alpha_{12}) \qquad (i=2,3)\,. \tag{C.24}$$

Once K_2 and K_3 are invariant under the twisting transformation in (4.15), from its definition in (4.10), it is immediate that T is also invariant under the same transformation.

C.2.3 Cyclic symmetry

It is easy to see that K_2 is not cyclic invariant. For this purpose it is enough to look at the first term in the α' expansion of it, in (4.3), which clearly does not respect the cyclic invariance. In this section we will prove that K_3 and T remain invariant under the following cyclic permutation of indexes:

$$(1,2,3,4,5) \rightarrow (2,3,4,5,1)\,. \tag{C.25}$$

So we need to prove that

$$K_3(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34}) \;=\; K_3(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45})\,, \tag{C.26}$$

$$T(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34}) \;=\; T(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45})\,. \tag{C.27}$$

- **Cyclic invariance of K_3.** Making the substitution

$$\begin{cases} x_2 \;=\; 1 - x_3{}' \\ x_3 \;=\; \dfrac{(1-x_3{}')}{(1-x_2{}')} \end{cases} \tag{C.28}$$

in the integral expression of K_3, in (C.6), it is not difficult to arrive to

$$K_3(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34}) = K_3(\alpha_{23};\alpha_{24};\alpha_{34};-\alpha_{13}-\alpha_{23}-\alpha_{34};\alpha_{12}+\alpha_{13}+\alpha_{23})\,. \tag{C.29}$$

Now, after considering the expressions for α_{13} and α_{35}, given in (C.12) and (C.16), this last relation becomes the one in (C.26).

- **Cyclic invariance of T.** The proof of the cyclic invariance in this case is very much more involved. We begin noticing that the desired condition (C.27) is equivalent to

$$\begin{aligned}\alpha_{23}\ \alpha_{45}\ K_2(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) \ &=\ \alpha_{12}\ \alpha_{34}\ K_2(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34})\ - \\ &\quad -(\alpha_{13}+\alpha_{23})\ \alpha_{24}\ K_3(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34})\,,\end{aligned} \tag{C.30}$$

once the definition of $T(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34})$, given in (4.10), together with the relations for the α_{ij}, given in (C.12)–(C.16), have been considered.

We will prove (C.30) in five steps:

1. Using the definition (C.5) we have that

$$\begin{aligned}K_2(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) = \int_0^1 dx_3{}' \int_0^{x_3{}'} & dx_2{}'\ x_3{}'^{2\alpha'\alpha_{24}}(1-x_3{}')^{2\alpha'\alpha_{45}-1}\times \\ & \times x_2{}'^{2\alpha'\alpha_{23}-1}(1-x_2{}')^{2\alpha'\alpha_{35}}\times \\ & \times\ (x_3{}'-x_2{}')^{2\alpha'\alpha_{34}}\,,\end{aligned} \tag{C.31}$$

   and making the substitution (C.28) for $K_2(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34})$ and $K_3(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34})$, given in (C.5) and (C.6), we have that

$$\begin{aligned}&K_2(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34}) = \\ &\quad = \int_0^1 dx_3{}' \int_0^{x_3{}'} dx_2{}'\ x_3{}'^{2\alpha'\alpha_{24}}(1-x_3{}')^{2\alpha'\alpha_{45}} x_2{}'^{2\alpha'\alpha_{23}}(1-x_2{}')^{2\alpha'\alpha_{35}-1}\times \\ &\qquad\qquad \times (x_3{}'-x_2{}')^{2\alpha'\alpha_{34}-1}\,,\end{aligned} \tag{C.32}$$

$$\begin{aligned}&K_3(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34}) = \\ &\quad = \int_0^1 dx_3{}' \int_0^{x_3{}'} dx_2{}'\ x_3{}'^{2\alpha'\alpha_{24}-1}(1-x_3{}')^{2\alpha'\alpha_{45}} x_2{}'^{2\alpha'\alpha_{23}}(1-x_2{}')^{2\alpha'\alpha_{35}-1}\times \\ &\qquad\qquad \times (x_3{}'-x_2{}')^{2\alpha'\alpha_{34}}\,.\end{aligned} \tag{C.33}$$

2. Doing integration by parts in (C.31), with respect to the $x_2{}'$ variable, it may be proved that

$$\begin{aligned}&K_2(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) = \\ &\quad = \frac{\alpha_{35}}{\alpha_{23}} M(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) + \frac{\alpha_{34}}{\alpha_{23}} N(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45})\,,\end{aligned} \tag{C.34}$$

   where

$$\begin{aligned}&M(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) = \\ &\quad = \int_0^1 dx_3{}' \int_0^{x_3{}'} dx_2{}'\ x_3{}'^{2\alpha'\alpha_{24}}(1-x_3{}')^{2\alpha'\alpha_{45}-1} x_2{}'^{2\alpha'\alpha_{23}}(1-x_2{}')^{2\alpha'\alpha_{35}-1}\times \\ &\qquad\qquad \times (x_3{}'-x_2{}')^{2\alpha'\alpha_{34}}\,,\end{aligned} \tag{C.35}$$

$$\begin{aligned}&N(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) = \\ &\quad = \int_0^1 dx_3{}' \int_0^{x_3{}'} dx_2{}'\ x_3{}'^{2\alpha'\alpha_{24}}(1-x_3{}')^{2\alpha'\alpha_{45}-1} x_2{}'^{2\alpha'\alpha_{23}}(1-x_2{}')^{2\alpha'\alpha_{35}}\times \\ &\qquad\qquad \times (x_3{}'-x_2{}')^{2\alpha'\alpha_{34}-1}\,.\end{aligned} \tag{C.36}$$

3. Now, doing integration by parts in (C.33), with respect to the $x_3{}'$ variable, it may also be proved that

$$K_3(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34}) = \\ = \frac{\alpha_{45}}{\alpha_{24}} M(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) + \frac{\alpha_{34}}{\alpha_{24}} K_2(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34})\,. \quad \text{(C.37)}$$

4. Noticing that

$$\frac{1}{(1-x_3{}')(x_3{}'-x_2{}')} = \frac{1}{(1-x_2{}')(1-x_3{}')} + \frac{1}{(1-x_2{}')(x_3{}'-x_2{}')}\,, \quad \text{(C.38)}$$

and using (C.32), (C.35) and (C.36), we have that

$$N(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) = \\ = M(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45}) + K_2(\alpha_{12};\alpha_{13};\alpha_{23};\alpha_{24};\alpha_{34})\,. \quad \text{(C.39)}$$

5. Finally, eliminating $M(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45})$ and $N(\alpha_{23};\alpha_{24};\alpha_{34};\alpha_{35};\alpha_{45})$ from equations (C.35), (C.37) and (C.39), leads precisely to (C.30).

Now that the proof of (C.26) and (C.27) has been done, then it may be repeated for the rest of the cyclic permutations, guaranteeing that K_3 and T remain invariant under them. We have also checked this.

C.2.4 The α' expansions of $(2\alpha')^2 K_3$ and T

A detailed calculation of the α' expansions of K_2 and K_3 was done in [42], up to $\mathcal{O}((2\alpha')^4)$ terms in both cases. This was done using Harmonic Polylogarithms [46] and the *harmpol* package of FORM [47] (see [34, appendix A.3] for more details about the type of calculations involved). The α' expansion of T was also obtained in [42], up to $\mathcal{O}((2\alpha')^6)$ terms, by directly using its definition (4.10) and the expansions of K_2 and K_3. In this section we will only write the α' expansions of $(2\alpha')^2 K_3$ and T.

It is not very difficult to prove that the α' expansions of $(2\alpha')^2 K_3$ and T are power series, that is, they do not have poles in α' (as K_2 does[28]). So, at every order in α', there only appear polynomial expressions which, after using the relations (C.12)–(C.16), may be written in terms of the five α_{ij} variables mentioned in section C.2.1. Each of these polynomial expressions respects the *twisting* and the *cyclic* symmetry, that were seen to be satisfied by $(2\alpha')^2 K_3$ and T in sections C.2.2 and C.2.3. The result obtained, written in terms of polynomial cyclic invariants (which will be specified in the next subsection), is the following [42]:

$$(2\alpha')^2 K_3 = (2\alpha')^2 \zeta(2) I_1^{(0)} - (2\alpha')^3 \zeta(3) I_1^{(1)} + (2\alpha')^4 \frac{2}{5}\zeta(2)^2 \left(I_1^{(2)} + \frac{1}{4} I_2^{(2)} + I_3^{(2)} \right) + \\ +(2\alpha')^5 \Big[-\zeta(5) \left(I_1^{(3)} + I_3^{(3)} + I_4^{(3)} \right) + (-2\zeta(5) + \zeta(2)\zeta(3)) \times$$

[28] See eq. (4.3).

$$\times\left(I_2^{(3)}+I_5^{(3)}+I_6^{(3)}\right)+\left(\frac{7}{2}\zeta(5)-2\zeta(2)\zeta(3)\right)I_7^{(3)}\Bigg]+$$
$$+(2\alpha')^6\Bigg[\frac{8}{35}\zeta(2)^3\left(I_1^{(4)}+I_3^{(4)}+I_4^{(4)}+I_7^{(4)}\right)+$$
$$+\left(\frac{6}{35}\zeta(2)^3-\frac{1}{2}\zeta(3)^2\right)\left(I_2^{(4)}+I_5^{(4)}+I_8^{(4)}-I_9^{(4)}-I_{12}^{(4)}+I_{13}^{(4)}\right)+$$
$$+\left(\frac{23}{70}\zeta(2)^3-\zeta(3)^2\right)\left(I_6^{(4)}+I_{10}^{(4)}\right)+\left(-\frac{26}{105}\zeta(2)^3+\zeta(3)^2\right)I_{11}^{(4)}+$$
$$+\left(-\frac{67}{105}\zeta(2)^3+2\zeta(3)^2\right)I_{14}^{(4)}\Bigg]+\mathcal{O}((2\alpha')^7\ ), \tag{C.40}$$

$$T=I_1^{(0)}-(2\alpha')^3\zeta(3)I_6^{(3)}+\ (2\alpha')^4\frac{2}{5}\zeta(2)^2\left(I_8^{(4)}+\frac{1}{4}I_{10}^{(4)}+I_{13}^{(4)}+I_{14}^{(4)}\right)+$$
$$+(2\alpha')^5\Bigg[-\zeta(5)\left(I_{10}^{(5)}+I_{15}^{(5)}+I_{19}^{(5)}+I_{22}^{(5)}+I_{25}^{(5)}\right)+$$
$$+(-2\zeta(5)+\zeta(2)\zeta(3))\left(I_{12}^{(5)}+I_{16}^{(5)}+I_{18}^{(5)}\right)+$$
$$+\left(\frac{7}{2}\zeta(5)-2\zeta(2)\zeta(3)\right)\left(I_{23}^{(5)}+I_{24}^{(5)}\right)\Bigg]+$$
$$+\ (2\alpha')^6\ \Bigg[\frac{8}{35}\zeta(2)^3\left(I_{10}^{(6)}+I_{15}^{(6)}+I_{21}^{(6)}+I_{26}^{(6)}+I_{30}^{(6)}+I_{33}^{(6)}+I_{41}^{(6)}\right)+$$
$$+\left(\frac{6}{35}\zeta(2)^3-\frac{1}{2}\zeta(3)^2\right)\times$$
$$\times\left(I_{12}^{(6)}+I_{18}^{(6)}+I_{29}^{(6)}-I_{31}^{(6)}-I_{32}^{(6)}+I_{34}^{(6)}-I_{36}^{(6)}-I_{38}^{(6)}\right)+$$
$$+\left(\frac{23}{70}\zeta(2)^3-\zeta(3)^2\right)\left(I_{20}^{(6)}+I_{27}^{(6)}\right)+$$
$$+\left(-\frac{26}{105}\zeta(2)^3+\zeta(3)^2\right)(I_{39}^{(6)}+I_{40}^{(6)})+$$
$$+\left(-\frac{67}{105}\zeta(2)^3+2\zeta(3)^2\right)I_{37}^{(6)}+$$
$$+\left(-\frac{109}{210}\zeta(2)^3-\frac{3}{2}\zeta(3)^2\right)I_{42}^{(6)}\Bigg]+\mathcal{O}((2\alpha')^7)\,. \tag{C.41}$$

Here $I_j^{(i)}$ denotes the j-th polynomial cyclic invariant of degree i.

C.2.5 Polynomial cyclic invariants

The polynomial cyclic invariants are uniquely determined up to a global factor which we have chosen to be 1. In the next lines we list them up to sixth degree. At degrees 0, 1, 2, 3, 4, 5 and 6 there are, respectively, one, one, three, seven, fourteen, twenty six and forty two linearly independent polynomial cyclic invariants.

- Degree 0:

$$I_1^{(0)}=1\,. \tag{C.42}$$

- Degree 1:

$$I_1^{(1)} = \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{45} + \alpha_{51}\,. \tag{C.43}$$

- Degree 2:

$$\begin{aligned}
I_1^{(2)} &= \alpha_{12}^2 + \alpha_{23}^2 + \alpha_{34}^2 + \alpha_{45}^2 + \alpha_{51}^2\,,\\
I_2^{(2)} &= \alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{34} + \alpha_{34}\alpha_{45} + \alpha_{45}\alpha_{51} + \alpha_{51}\alpha_{12}\,,\\
I_3^{(2)} &= \alpha_{12}\alpha_{34} + \alpha_{23}\alpha_{45} + \alpha_{34}\alpha_{51} + \alpha_{45}\alpha_{12} + \alpha_{51}\alpha_{23}\,.
\end{aligned} \tag{C.44}$$

- Degree 3:

$$\begin{aligned}
I_1^{(3)} &= \alpha_{12}^3 + \alpha_{23}^3 + \alpha_{34}^3 + \alpha_{45}^3 + \alpha_{51}^3\,,\\
I_2^{(3)} &= \alpha_{12}^2\alpha_{23} + \alpha_{23}^2\alpha_{34} + \alpha_{34}^2\alpha_{45} + \alpha_{45}^2\alpha_{51} + \alpha_{51}^2\alpha_{12}\,,\\
I_3^{(3)} &= \alpha_{12}^2\alpha_{34} + \alpha_{23}^2\alpha_{45} + \alpha_{34}^2\alpha_{51} + \alpha_{45}^2\alpha_{12} + \alpha_{51}^2\alpha_{23}\,,\\
I_4^{(3)} &= \alpha_{12}^2\alpha_{45} + \alpha_{23}^2\alpha_{51} + \alpha_{34}^2\alpha_{12} + \alpha_{45}^2\alpha_{23} + \alpha_{51}^2\alpha_{34}\,,\\
I_5^{(3)} &= \alpha_{12}^2\alpha_{51} + \alpha_{23}^2\alpha_{12} + \alpha_{34}^2\alpha_{23} + \alpha_{45}^2\alpha_{34} + \alpha_{51}^2\alpha_{45}\,,\\
I_6^{(3)} &= \alpha_{12}\alpha_{23}\alpha_{34} + \alpha_{23}\alpha_{34}\alpha_{45} + \alpha_{34}\alpha_{45}\alpha_{51} + \alpha_{45}\alpha_{51}\alpha_{12} + \alpha_{51}\alpha_{12}\alpha_{23}\,,\\
I_7^{(3)} &= \alpha_{12}\alpha_{34}\alpha_{45} + \alpha_{23}\alpha_{45}\alpha_{51} + \alpha_{34}\alpha_{51}\alpha_{12} + \alpha_{45}\alpha_{12}\alpha_{23} + \alpha_{51}\alpha_{23}\alpha_{34}\,.
\end{aligned} \tag{C.45}$$

- Degree 4:

$$\begin{aligned}
I_1^{(4)} &= \alpha_{12}^4 + \alpha_{23}^4 + \alpha_{34}^4 + \alpha_{45}^4 + \alpha_{51}^4\,,\\
I_2^{(4)} &= \alpha_{12}^3\alpha_{23} + \alpha_{23}^3\alpha_{34} + \alpha_{34}^3\alpha_{45} + \alpha_{45}^3\alpha_{51} + \alpha_{51}^3\alpha_{12}\,,\\
I_3^{(4)} &= \alpha_{12}^3\alpha_{34} + \alpha_{23}^3\alpha_{45} + \alpha_{34}^3\alpha_{51} + \alpha_{45}^3\alpha_{12} + \alpha_{51}^3\alpha_{23}\,.\\
I_4^{(4)} &= \alpha_{12}^3\alpha_{45} + \alpha_{23}^3\alpha_{51} + \alpha_{34}^3\alpha_{12} + \alpha_{45}^3\alpha_{23} + \alpha_{51}^3\alpha_{34}\,.\\
I_5^{(4)} &= \alpha_{12}^3\alpha_{51} + \alpha_{23}^3\alpha_{12} + \alpha_{34}^3\alpha_{23} + \alpha_{45}^3\alpha_{34} + \alpha_{51}^3\alpha_{45}\,.\\
I_6^{(4)} &= \alpha_{12}^2\alpha_{23}^2 + \alpha_{23}^2\alpha_{34}^2 + \alpha_{34}^2\alpha_{45}^2 + \alpha_{45}^2\alpha_{51}^2 + \alpha_{51}^2\alpha_{12}^2\,,\\
I_7^{(4)} &= \alpha_{12}^2\alpha_{34}^2 + \alpha_{23}^2\alpha_{45}^2 + \alpha_{34}^2\alpha_{51}^2 + \alpha_{45}^2\alpha_{12}^2 + \alpha_{51}^2\alpha_{23}^2\,,\\
I_8^{(4)} &= \alpha_{12}^2\alpha_{23}\alpha_{34} + \alpha_{23}^2\alpha_{34}\alpha_{45} + \alpha_{34}^2\alpha_{45}\alpha_{51} + \alpha_{45}^2\alpha_{51}\alpha_{12} + \alpha_{51}^2\alpha_{12}\alpha_{23}\,,\\
I_9^{(4)} &= \alpha_{12}^2\alpha_{23}\alpha_{45} + \alpha_{23}^2\alpha_{34}\alpha_{51} + \alpha_{34}^2\alpha_{45}\alpha_{12} + \alpha_{45}^2\alpha_{51}\alpha_{23} + \alpha_{51}^2\alpha_{12}\alpha_{34}\,,\\
I_{10}^{(4)} &= \alpha_{12}^2\alpha_{23}\alpha_{51} + \alpha_{23}^2\alpha_{34}\alpha_{12} + \alpha_{34}^2\alpha_{45}\alpha_{23} + \alpha_{45}^2\alpha_{51}\alpha_{34} + \alpha_{51}^2\alpha_{12}\alpha_{45}\,,\\
I_{11}^{(4)} &= \alpha_{12}^2\alpha_{34}\alpha_{45} + \alpha_{23}^2\alpha_{45}\alpha_{51} + \alpha_{34}^2\alpha_{51}\alpha_{12} + \alpha_{45}^2\alpha_{12}\alpha_{23} + \alpha_{51}^2\alpha_{23}\alpha_{34}\,,\\
I_{12}^{(4)} &= \alpha_{12}^2\alpha_{34}\alpha_{51} + \alpha_{23}^2\alpha_{45}\alpha_{12} + \alpha_{34}^2\alpha_{51}\alpha_{23} + \alpha_{45}^2\alpha_{12}\alpha_{34} + \alpha_{51}^2\alpha_{23}\alpha_{45}\,,\\
I_{13}^{(4)} &= \alpha_{12}^2\alpha_{45}\alpha_{51} + \alpha_{23}^2\alpha_{51}\alpha_{12} + \alpha_{34}^2\alpha_{12}\alpha_{23} + \alpha_{45}^2\alpha_{23}\alpha_{34} + \alpha_{51}^2\alpha_{34}\alpha_{45}\,,\\
I_{14}^{(4)} &= \alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45}+\alpha_{23}\alpha_{34}\alpha_{45}\alpha_{51}+\alpha_{34}\alpha_{45}\alpha_{51}\alpha_{12}+\alpha_{45}\alpha_{51}\alpha_{12}\alpha_{23}+\alpha_{51}\alpha_{12}\alpha_{23}\alpha_{34}\,.
\end{aligned} \tag{C.46}$$

- Degree 5:

$$\begin{aligned}
I_1^{(5)} &= \alpha_{12}^5 + \alpha_{23}^5 + \alpha_{34}^5 + \alpha_{45}^5 + \alpha_{51}^5\,,\\
I_2^{(5)} &= \alpha_{12}^4\alpha_{23} + \alpha_{23}^4\alpha_{34} + \alpha_{34}^4\alpha_{45} + \alpha_{45}^4\alpha_{51} + \alpha_{51}^4\alpha_{12}\,,
\end{aligned}$$

$$
\begin{aligned}
I^{(5)}_3 &= \alpha_{12}^4\alpha_{34} + \alpha_{23}^4\alpha_{45} + \alpha_{34}^4\alpha_{51} + \alpha_{45}^4\alpha_{12} + \alpha_{51}^4\alpha_{23}\,,\\
I^{(5)}_4 &= \alpha_{12}^4\alpha_{45} + \alpha_{23}^4\alpha_{51} + \alpha_{34}^4\alpha_{12} + \alpha_{45}^4\alpha_{23} + \alpha_{51}^4\alpha_{34}\,,\\
I^{(5)}_5 &= \alpha_{12}^4\alpha_{51} + \alpha_{23}^4\alpha_{12} + \alpha_{34}^4\alpha_{23} + \alpha_{45}^4\alpha_{34} + \alpha_{51}^4\alpha_{45}\,,\\
I^{(5)}_6 &= \alpha_{12}^3\alpha_{23}^2 + \alpha_{23}^3\alpha_{34}^2 + \alpha_{34}^3\alpha_{45}^2 + \alpha_{45}^3\alpha_{51}^2 + \alpha_{51}^3\alpha_{12}^2\,,\\
I^{(5)}_7 &= \alpha_{12}^3\alpha_{34}^2 + \alpha_{23}^3\alpha_{45}^2 + \alpha_{34}^3\alpha_{51}^2 + \alpha_{45}^3\alpha_{12}^2 + \alpha_{51}^3\alpha_{23}^2\,,\\
I^{(5)}_8 &= \alpha_{12}^3\alpha_{45}^2 + \alpha_{23}^3\alpha_{51}^2 + \alpha_{34}^3\alpha_{12}^2 + \alpha_{45}^3\alpha_{23}^2 + \alpha_{51}^3\alpha_{34}^2\,,\\
I^{(5)}_9 &= \alpha_{12}^3\alpha_{51}^2 + \alpha_{23}^3\alpha_{12}^2 + \alpha_{34}^3\alpha_{23}^2 + \alpha_{45}^3\alpha_{34}^2 + \alpha_{51}^3\alpha_{45}^2\,,\\
I^{(5)}_{10} &= \alpha_{12}^3\alpha_{23}\alpha_{34} + \alpha_{23}^3\alpha_{34}\alpha_{45} + \alpha_{34}^3\alpha_{45}\alpha_{51} + \alpha_{45}^3\alpha_{51}\alpha_{12} + \alpha_{51}^3\alpha_{12}\alpha_{23}\,,\\
I^{(5)}_{11} &= \alpha_{12}^3\alpha_{23}\alpha_{45} + \alpha_{23}^3\alpha_{34}\alpha_{51} + \alpha_{34}^3\alpha_{45}\alpha_{12} + \alpha_{45}^3\alpha_{51}\alpha_{23} + \alpha_{51}^3\alpha_{12}\alpha_{34}\,,\\
I^{5)}_{12} &= \alpha_{12}^3\alpha_{23}\alpha_{51} + \alpha_{23}^3\alpha_{34}\alpha_{12} + \alpha_{34}^3\alpha_{45}\alpha_{23} + \alpha_{45}^3\alpha_{51}\alpha_{34} + \alpha_{51}^3\alpha_{12}\alpha_{45}\,,\\
I^{(5)}_{13} &= \alpha_{12}^3\alpha_{34}\alpha_{45} + \alpha_{23}^3\alpha_{45}\alpha_{51} + \alpha_{34}^3\alpha_{51}\alpha_{12} + \alpha_{45}^3\alpha_{12}\alpha_{23} + \alpha_{51}^3\alpha_{23}\alpha_{34}\,,\\
I^{(5)}_{14} &= \alpha_{12}^3\alpha_{34}\alpha_{51} + \alpha_{23}^3\alpha_{45}\alpha_{12} + \alpha_{34}^3\alpha_{51}\alpha_{23} + \alpha_{45}^3\alpha_{12}\alpha_{34} + \alpha_{51}^3\alpha_{23}\alpha_{45}\,,\\
I^{(5)}_{15} &= \alpha_{12}^3\alpha_{45}\alpha_{51} + \alpha_{23}^3\alpha_{51}\alpha_{12} + \alpha_{34}^3\alpha_{12}\alpha_{23} + \alpha_{45}^3\alpha_{23}\alpha_{34} + \alpha_{51}^3\alpha_{34}\alpha_{45}\,,\\
I^{(5)}_{16} &= \alpha_{12}^2\alpha_{23}^2\alpha_{34} + \alpha_{23}^2\alpha_{34}^2\alpha_{45} + \alpha_{34}^2\alpha_{45}^2\alpha_{51} + \alpha_{45}^2\alpha_{51}^2\alpha_{12} + \alpha_{51}^2\alpha_{12}^2\alpha_{23}\,,\\
I^{(5)}_{17} &= \alpha_{12}^2\alpha_{23}^2\alpha_{45} + \alpha_{23}^2\alpha_{34}^2\alpha_{51} + \alpha_{34}^2\alpha_{45}^2\alpha_{12} + \alpha_{45}^2\alpha_{51}^2\alpha_{23} + \alpha_{51}^2\alpha_{12}^2\alpha_{34}\,,\\
I^{(5)}_{18} &= \alpha_{12}^2\alpha_{23}^2\alpha_{51} + \alpha_{23}^2\alpha_{34}^2\alpha_{12} + \alpha_{34}^2\alpha_{45}^2\alpha_{23} + \alpha_{45}^2\alpha_{51}^2\alpha_{34} + \alpha_{51}^2\alpha_{12}^2\alpha_{45}\,,\\
I^{(5)}_{19} &= \alpha_{12}^2\alpha_{34}^2\alpha_{23} + \alpha_{23}^2\alpha_{45}^2\alpha_{34} + \alpha_{34}^2\alpha_{51}^2\alpha_{45} + \alpha_{45}^2\alpha_{12}^2\alpha_{51} + \alpha_{51}^2\alpha_{23}^2\alpha_{12}\,,\\
I^{(5)}_{20} &= \alpha_{12}^2\alpha_{34}^2\alpha_{45} + \alpha_{23}^2\alpha_{45}^2\alpha_{51} + \alpha_{34}^2\alpha_{51}^2\alpha_{12} + \alpha_{45}^2\alpha_{12}^2\alpha_{23} + \alpha_{51}^2\alpha_{23}^2\alpha_{34}\,,\\
I^{(5)}_{21} &= \alpha_{12}^2\alpha_{34}^2\alpha_{51} + \alpha_{23}^2\alpha_{45}^2\alpha_{12} + \alpha_{34}^2\alpha_{51}^2\alpha_{23} + \alpha_{45}^2\alpha_{12}^2\alpha_{34} + \alpha_{51}^2\alpha_{23}^2\alpha_{45}\,,\\
I^{(5)}_{22} &= \alpha_{12}^2\alpha_{23}\alpha_{34}\alpha_{45} + \alpha_{23}^2\alpha_{34}\alpha_{45}\alpha_{51} + \alpha_{34}^2\alpha_{45}\alpha_{51}\alpha_{12} + \alpha_{45}^2\alpha_{51}\alpha_{12}\alpha_{23} +\\
&\quad + \alpha_{51}^2\alpha_{12}\alpha_{23}\alpha_{34}\,,\\
I^{(5)}_{23} &= \alpha_{12}^2\alpha_{23}\alpha_{34}\alpha_{51} + \alpha_{23}^2\alpha_{34}\alpha_{45}\alpha_{12} + \alpha_{34}^2\alpha_{45}\alpha_{51}\alpha_{23} + \alpha_{45}^2\alpha_{51}\alpha_{12}\alpha_{34} +\\
&\quad + \alpha_{51}^2\alpha_{12}\alpha_{23}\alpha_{45}\,,\\
I^{(5)}_{24} &= \alpha_{12}^2\alpha_{23}\alpha_{45}\alpha_{51} + \alpha_{23}^2\alpha_{34}\alpha_{51}\alpha_{12} + \alpha_{34}^2\alpha_{45}\alpha_{12}\alpha_{23} + \alpha_{45}^2\alpha_{51}\alpha_{23}\alpha_{34} +\\
&\quad + \alpha_{51}^2\alpha_{12}\alpha_{34}\alpha_{45}\,,\\
I^{(5)}_{25} &= \alpha_{12}^2\alpha_{34}\alpha_{45}\alpha_{51} + \alpha_{23}^2\alpha_{45}\alpha_{51}\alpha_{12} + \alpha_{34}^2\alpha_{51}\alpha_{12}\alpha_{23} + \alpha_{45}^2\alpha_{12}\alpha_{23}\alpha_{34} +\\
&\quad + \alpha_{51}^2\alpha_{23}\alpha_{34}\alpha_{45}\,,\\
I^{(5)}_{26} &= \alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45}\alpha_{51}\,.
\end{aligned}
\tag{C.47}
$$

- Degree 6:

$$
\begin{aligned}
I^{(6)}_1 &= \alpha_{12}^6 + \alpha_{23}^6 + \alpha_{34}^6 + \alpha_{45}^6 + \alpha_{51}^6\,,\\
I^{(6)}_2 &= \alpha_{12}^5\alpha_{23} + \alpha_{23}^5\alpha_{34} + \alpha_{34}^5\alpha_{45} + \alpha_{45}^5\alpha_{51} + \alpha_{51}^5\alpha_{12}\,,\\
I^{(6)}_3 &= \alpha_{12}^5\alpha_{34} + \alpha_{23}^5\alpha_{45} + \alpha_{34}^5\alpha_{51} + \alpha_{45}^5\alpha_{12} + \alpha_{51}^5\alpha_{23}\,,\\
I^{(6)}_4 &= \alpha_{12}^5\alpha_{45} + \alpha_{23}^5\alpha_{51} + \alpha_{34}^5\alpha_{12} + \alpha_{45}^5\alpha_{23} + \alpha_{51}^5\alpha_{34}\,,\\
I^{(6)}_5 &= \alpha_{12}^5\alpha_{51} + \alpha_{23}^5\alpha_{12} + \alpha_{34}^5\alpha_{23} + \alpha_{45}^5\alpha_{34} + \alpha_{51}^5\alpha_{45}\,.\\
I^{(6)}_6 &= \alpha_{12}^4\alpha_{23}^2 + \alpha_{23}^4\alpha_{34}^2 + \alpha_{34}^4\alpha_{45}^2 + \alpha_{45}^4\alpha_{51}^2 + \alpha_{51}^4\alpha_{12}^2\,,
\end{aligned}
$$

$$
\begin{aligned}
I_7^{(6)} &= \alpha_{12}^4\alpha_{34}^2 + \alpha_{23}^4\alpha_{45}^2 + \alpha_{34}^4\alpha_{51}^2 + \alpha_{45}^4\alpha_{12}^2 + \alpha_{51}^4\alpha_{23}^2\,,\\
I_8^{(6)} &= \alpha_{12}^4\alpha_{45}^2 + \alpha_{23}^4\alpha_{51}^2 + \alpha_{34}^4\alpha_{12}^2 + \alpha_{45}^4\alpha_{23}^2 + \alpha_{51}^4\alpha_{34}^2\,,\\
I_9^{(6)} &= \alpha_{12}^4\alpha_{51}^2 + \alpha_{23}^4\alpha_{12}^2 + \alpha_{34}^4\alpha_{23}^2 + \alpha_{45}^4\alpha_{34}^2 + \alpha_{51}^4\alpha_{45}^2\,,\\
I_{10}^{(6)} &= \alpha_{12}^4\alpha_{23}\alpha_{34} + \alpha_{23}^4\alpha_{34}\alpha_{45} + \alpha_{34}^4\alpha_{45}\alpha_{51} + \alpha_{45}^4\alpha_{51}\alpha_{12} + \alpha_{51}^4\alpha_{12}\alpha_{23}\,,\\
I_{11}^{(6)} &= \alpha_{12}^4\alpha_{23}\alpha_{45} + \alpha_{23}^4\alpha_{34}\alpha_{51} + \alpha_{34}^4\alpha_{45}\alpha_{12} + \alpha_{45}^4\alpha_{51}\alpha_{23} + \alpha_{51}^4\alpha_{12}\alpha_{34}\,,\\
I_{12}^{(6)} &= \alpha_{12}^4\alpha_{23}\alpha_{51} + \alpha_{23}^4\alpha_{34}\alpha_{12} + \alpha_{34}^4\alpha_{45}\alpha_{23} + \alpha_{45}^4\alpha_{51}\alpha_{34} + \alpha_{51}^4\alpha_{12}\alpha_{45}\,,\\
I_{13}^{(6)} &= \alpha_{12}^4\alpha_{34}\alpha_{45} + \alpha_{23}^4\alpha_{45}\alpha_{51} + \alpha_{34}^4\alpha_{51}\alpha_{12} + \alpha_{45}^4\alpha_{12}\alpha_{23} + \alpha_{51}^4\alpha_{23}\alpha_{34}\,,\\
I_{14}^{(6)} &= \alpha_{12}^4\alpha_{34}\alpha_{51} + \alpha_{23}^4\alpha_{45}\alpha_{12} + \alpha_{34}^4\alpha_{51}\alpha_{23} + \alpha_{45}^4\alpha_{12}\alpha_{34} + \alpha_{51}^4\alpha_{23}\alpha_{45}\,,\\
I_{15}^{(6)} &= \alpha_{12}^4\alpha_{45}\alpha_{51} + \alpha_{23}^4\alpha_{51}\alpha_{12} + \alpha_{34}^4\alpha_{12}\alpha_{23} + \alpha_{45}^4\alpha_{23}\alpha_{34} + \alpha_{51}^4\alpha_{34}\alpha_{45}\,,\\
I_{16}^{(6)} &= \alpha_{12}^3\alpha_{23}^3 + \alpha_{23}^3\alpha_{34}^3 + \alpha_{34}^3\alpha_{45}^3 + \alpha_{45}^3\alpha_{51}^3 + \alpha_{51}^3\alpha_{12}^3\,,\\
I_{17}^{(6)} &= \alpha_{12}^3\alpha_{34}^3 + \alpha_{23}^3\alpha_{45}^3 + \alpha_{34}^3\alpha_{51}^3 + \alpha_{45}^3\alpha_{12}^3 + \alpha_{51}^3\alpha_{23}^3\,,\\
I_{18}^{(6)} &= \alpha_{12}^3\alpha_{23}^2\alpha_{34} + \alpha_{23}^3\alpha_{34}^2\alpha_{45} + \alpha_{34}^3\alpha_{45}^2\alpha_{51} + \alpha_{45}^3\alpha_{51}^2\alpha_{12} + \alpha_{51}^3\alpha_{12}^2\alpha_{23}\,,\\
I_{19}^{(6)} &= \alpha_{12}^3\alpha_{23}^2\alpha_{45} + \alpha_{23}^3\alpha_{34}^2\alpha_{51} + \alpha_{34}^3\alpha_{45}^2\alpha_{12} + \alpha_{45}^3\alpha_{51}^2\alpha_{23} + \alpha_{51}^3\alpha_{12}^2\alpha_{34}\,,\\
I_{20}^{(6)} &= \alpha_{12}^3\alpha_{23}^2\alpha_{51} + \alpha_{23}^3\alpha_{34}^2\alpha_{12} + \alpha_{34}^3\alpha_{45}^2\alpha_{23} + \alpha_{45}^3\alpha_{51}^2\alpha_{34} + \alpha_{51}^3\alpha_{12}^2\alpha_{45}\,,\\
I_{21}^{(6)} &= \alpha_{12}^3\alpha_{34}^2\alpha_{23} + \alpha_{23}^3\alpha_{45}^2\alpha_{34} + \alpha_{34}^3\alpha_{51}^2\alpha_{45} + \alpha_{45}^3\alpha_{12}^2\alpha_{51} + \alpha_{51}^3\alpha_{23}^2\alpha_{12}\,,\\
I_{22}^{(6)} &= \alpha_{12}^3\alpha_{34}^2\alpha_{45} + \alpha_{23}^3\alpha_{45}^2\alpha_{51} + \alpha_{34}^3\alpha_{51}^2\alpha_{12} + \alpha_{45}^3\alpha_{12}^2\alpha_{23} + \alpha_{51}^3\alpha_{23}^2\alpha_{34}\,,\\
I_{23}^{(6)} &= \alpha_{12}^3\alpha_{34}^2\alpha_{51} + \alpha_{23}^3\alpha_{45}^2\alpha_{12} + \alpha_{34}^3\alpha_{51}^2\alpha_{23} + \alpha_{45}^3\alpha_{12}^2\alpha_{34} + \alpha_{51}^3\alpha_{23}^2\alpha_{45}\,,\\
I_{24}^{(6)} &= \alpha_{12}^3\alpha_{45}^2\alpha_{23} + \alpha_{23}^3\alpha_{51}^2\alpha_{34} + \alpha_{34}^3\alpha_{12}^2\alpha_{45} + \alpha_{45}^3\alpha_{23}^2\alpha_{51} + \alpha_{51}^3\alpha_{34}^2\alpha_{12}\,,\\
I_{25}^{(6)} &= \alpha_{12}^3\alpha_{45}^2\alpha_{34} + \alpha_{23}^3\alpha_{51}^2\alpha_{45} + \alpha_{34}^3\alpha_{12}^2\alpha_{51} + \alpha_{45}^3\alpha_{23}^2\alpha_{12} + \alpha_{51}^3\alpha_{34}^2\alpha_{23}\,,\\
I_{26}^{(6)} &= \alpha_{12}^3\alpha_{45}^2\alpha_{51} + \alpha_{23}^3\alpha_{51}^2\alpha_{12} + \alpha_{34}^3\alpha_{12}^2\alpha_{23} + \alpha_{45}^3\alpha_{23}^2\alpha_{34} + \alpha_{51}^3\alpha_{34}^2\alpha_{45}\,,\\
I_{27}^{(6)} &= \alpha_{12}^3\alpha_{51}^2\alpha_{23} + \alpha_{23}^3\alpha_{12}^2\alpha_{34} + \alpha_{34}^3\alpha_{23}^2\alpha_{45} + \alpha_{45}^3\alpha_{34}^2\alpha_{51} + \alpha_{51}^3\alpha_{45}^2\alpha_{12}\,,\\
I_{28}^{(6)} &= \alpha_{12}^3\alpha_{51}^2\alpha_{34} + \alpha_{23}^3\alpha_{12}^2\alpha_{45} + \alpha_{34}^3\alpha_{23}^2\alpha_{51} + \alpha_{45}^3\alpha_{34}^2\alpha_{12} + \alpha_{51}^3\alpha_{45}^2\alpha_{23}\,,\\
I_{29}^{(6)} &= \alpha_{12}^3\alpha_{51}^2\alpha_{45} + \alpha_{23}^3\alpha_{12}^2\alpha_{51} + \alpha_{34}^3\alpha_{23}^2\alpha_{12} + \alpha_{45}^3\alpha_{34}^2\alpha_{23} + \alpha_{51}^3\alpha_{45}^2\alpha_{34}\,,\\
I_{30}^{(6)} &= \alpha_{12}^3\alpha_{23}\alpha_{34}\alpha_{45} + \alpha_{23}^3\alpha_{34}\alpha_{45}\alpha_{51} + \alpha_{34}^3\alpha_{45}\alpha_{51}\alpha_{12} + \alpha_{45}^3\alpha_{51}\alpha_{12}\alpha_{23} +\\
&\quad + \alpha_{51}^3\alpha_{12}\alpha_{23}\alpha_{34}\,,\\
I_{31}^{(6)} &= \alpha_{12}^3\alpha_{23}\alpha_{34}\alpha_{51} + \alpha_{23}^3\alpha_{34}\alpha_{45}\alpha_{12} + \alpha_{34}^3\alpha_{45}\alpha_{51}\alpha_{23} + \alpha_{45}^3\alpha_{51}\alpha_{12}\alpha_{34} +\\
&\quad + \alpha_{51}^3\alpha_{12}\alpha_{23}\alpha_{45}\,,\\
I_{32}^{(6)} &= \alpha_{12}^3\alpha_{23}\alpha_{45}\alpha_{51} + \alpha_{23}^3\alpha_{34}\alpha_{51}\alpha_{12} + \alpha_{34}^3\alpha_{45}\alpha_{12}\alpha_{23} + \alpha_{45}^3\alpha_{51}\alpha_{23}\alpha_{34} +\\
&\quad + \alpha_{51}^3\alpha_{12}\alpha_{34}\alpha_{45}\,,\\
I_{33}^{(6)} &= \alpha_{12}^3\alpha_{34}\alpha_{45}\alpha_{51} + \alpha_{23}^3\alpha_{45}\alpha_{51}\alpha_{12} + \alpha_{34}^3\alpha_{51}\alpha_{12}\alpha_{23} + \alpha_{45}^3\alpha_{12}\alpha_{23}\alpha_{34} +\\
&\quad + \alpha_{51}^3\alpha_{23}\alpha_{34}\alpha_{45}\,,\\
I_{34}^{(6)} &= \alpha_{12}^2\alpha_{23}^2\alpha_{34}^2 + \alpha_{23}^2\alpha_{34}^2\alpha_{45}^2 + \alpha_{34}^2\alpha_{45}^2\alpha_{51}^2 + \alpha_{45}^2\alpha_{51}^2\alpha_{12}^2 + \alpha_{51}^2\alpha_{12}^2\alpha_{23}^2\,,\\
I_{35}^{(6)} &= \alpha_{12}^2\alpha_{23}^2\alpha_{45}^2 + \alpha_{23}^2\alpha_{34}^2\alpha_{51}^2 + \alpha_{34}^2\alpha_{45}^2\alpha_{12}^2 + \alpha_{45}^2\alpha_{51}^2\alpha_{23}^2 + \alpha_{51}^2\alpha_{12}^2\alpha_{34}^2\,,\\
I_{36}^{(6)} &= \alpha_{12}^2\alpha_{23}^2\alpha_{34}\alpha_{45} + \alpha_{23}^2\alpha_{34}^2\alpha_{45}\alpha_{51} + \alpha_{34}^2\alpha_{45}^2\alpha_{51}\alpha_{12} + \alpha_{45}^2\alpha_{51}^2\alpha_{12}\alpha_{23} +
\end{aligned}
$$

$$
\begin{aligned}
& + \alpha_{51}^2\alpha_{12}^2\alpha_{23}\alpha_{34}\,, \\
I^{(6)}_{37} &= \alpha_{12}^2\alpha_{23}^2\alpha_{34}\alpha_{51} + \alpha_{23}^2\alpha_{34}^2\alpha_{45}\alpha_{12} + \alpha_{34}^2\alpha_{45}^2\alpha_{51}\alpha_{23} + \alpha_{45}^2\alpha_{51}^2\alpha_{12}\alpha_{34} + \\
& + \alpha_{51}^2\alpha_{12}^2\alpha_{23}\alpha_{45}\,, \\
I^{(6)}_{38} &= \alpha_{12}^2\alpha_{23}^2\alpha_{45}\alpha_{51} + \alpha_{23}^2\alpha_{34}^2\alpha_{51}\alpha_{12} + \alpha_{34}^2\alpha_{45}^2\alpha_{12}\alpha_{23} + \alpha_{45}^2\alpha_{51}^2\alpha_{23}\alpha_{34} + \\
& + \alpha_{51}^2\alpha_{12}^2\alpha_{34}\alpha_{45}\,, \\
I^{(6)}_{39} &= \alpha_{12}^2\alpha_{34}^2\alpha_{23}\alpha_{45} + \alpha_{23}^2\alpha_{45}^2\alpha_{34}\alpha_{51} + \alpha_{34}^2\alpha_{51}^2\alpha_{45}\alpha_{12} + \alpha_{45}^2\alpha_{12}^2\alpha_{51}\alpha_{23} + \\
& + \alpha_{51}^2\alpha_{23}^2\alpha_{12}\alpha_{34}\,, \\
I^{(6)}_{40} &= \alpha_{12}^2\alpha_{34}^2\alpha_{23}\alpha_{51} + \alpha_{23}^2\alpha_{45}^2\alpha_{34}\alpha_{12} + \alpha_{34}^2\alpha_{51}^2\alpha_{45}\alpha_{23} + \alpha_{45}^2\alpha_{12}^2\alpha_{51}\alpha_{34} + \\
& + \alpha_{51}^2\alpha_{23}^2\alpha_{12}\alpha_{45}\,, \\
I^{(6)}_{41} &= \alpha_{12}^2\alpha_{34}^2\alpha_{45}\alpha_{51} + \alpha_{23}^2\alpha_{45}^2\alpha_{51}\alpha_{12} + \alpha_{34}^2\alpha_{51}^2\alpha_{12}\alpha_{23} + \alpha_{45}^2\alpha_{12}^2\alpha_{23}\alpha_{34} + \\
& + \alpha_{51}^2\alpha_{23}^2\alpha_{34}\alpha_{45}\,, \\
I^{(6)}_{42} &= \alpha_{12}^2\alpha_{23}\alpha_{34}\alpha_{45}\alpha_{51} + \alpha_{23}^2\alpha_{34}\alpha_{45}\alpha_{51}\alpha_{12} + \alpha_{34}^2\alpha_{45}\alpha_{51}\alpha_{12}\alpha_{23} + \\
& + \alpha_{45}^2\alpha_{51}\alpha_{12}\alpha_{23}\alpha_{34} + \alpha_{51}^2\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45}\,. \qquad \text{(C.48)}
\end{aligned}
$$

C.3 $H^{(1)}$, $P^{(1)}$, $U^{(1)}$, $W^{(1)}$, $Z^{(1)}$ and Δ factors

The α' dependent factors of formula (5.2), except for $Z^{(1)}$, are all given explicitly in terms of the function f of (3.8), the factor K_3 of (C.6) and the factor T of (4.10):[29]

$$
H^{(1)} = -(2\alpha')^2 \frac{f(-2\alpha_{23},-2\alpha_{34}) - f(-2\alpha_{34},-2\alpha_{45})}{\alpha_{23}-\alpha_{45}}\,, \tag{C.49}
$$

$$
P^{(1)} = (2\alpha')^2 \frac{K_3 - K_3\Big|_{\alpha_{34}=0}}{\alpha_{34}}\,, \tag{C.50}
$$

$$
U^{(1)} = \frac{(2\alpha')^2\left\{ f(-2\alpha_{51},-2\alpha_{12}) + K_3\Big|_{\alpha_{34}=0}\right\} - \alpha_{45}H^{(3)} - \alpha_{23}H^{(4)}}{\alpha_{23}\alpha_{45}}\,, \tag{C.51}
$$

$$
W^{(1)} = \frac{(2\alpha')^2\left[K_3 - K_3\Big|_{\alpha_{51}=0} - K_3\Big|_{\alpha_{12}=0}\right] + \alpha_{23}H^{(1)} - (2\alpha')^2 f(-2\alpha_{34},-2\alpha_{45})}{\alpha_{51}\alpha_{12}}\,, \tag{C.52}
$$

$$
\begin{aligned}
Z^{(1)} &= (2\alpha')^2 \times \\
&\times \frac{G(-2\alpha_{34},-2\alpha_{45},-\alpha_{25}-\alpha_{34},-\alpha_{23}-\alpha_{45}) - G(-2\alpha_{34},-2\alpha_{23},-\alpha_{25}-\alpha_{34},-\alpha_{23}-\alpha_{45})}{\alpha_{23}-\alpha_{45}}\,,
\end{aligned} \tag{C.53}
$$

$$
\Delta = \frac{T - 1 - \left[\left\{ T\Big|_{\alpha_{23}=0} - 1 - \alpha_{12}\alpha_{23}\alpha_{34}H^{(5)} \right\} + (\text{cyclic permutations}) \right]}{\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45}\alpha_{51}}\,, \tag{C.54}
$$

[29] In this formula, the factors $H^{(j)}$ (with $j = 2,3,4,5$) are constructed by means of cyclic permutations of the factor $H^{(1)}$. For example, to obtain the expression for $H^{(2)}$ we make the following change of indexes in the α_{ij} variables: $(1,2,3,4,5) \to (2,3,4,5,1)$.

where the function $G(a,b,c,d)$ that appears in (C.53) is defined by the α' series

$$
\begin{aligned}
G(a,b,c,d) &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{a_{mn}\alpha'^{m+n}}{\begin{pmatrix} m+n \\ m \end{pmatrix}} \times \\
&\quad \times \sum_{p=0}^{m-1}\sum_{q=0}^{n} \begin{pmatrix} m-1-p+n-q \\ m-1-p \end{pmatrix} \begin{pmatrix} p+q \\ p \end{pmatrix} a^{m-1-p} b^{n-q} c^{p} d^{q} \,. \qquad \text{(C.55)}
\end{aligned}
$$

Here, the (completely known) coefficients a_{mn} are the ones that appear in the expansion of the function $f(s,t)$, in (C.4). May be it is possible to write the function $G(a,b,c,d)$ in a closed form (in terms of Hypergeometric functions, for example), but we have not succeeded in doing so.

$G(a,b,c,d)$ begins like

$$
\begin{aligned}
G(a,b,c,d) &= -\zeta(3)\alpha' - \frac{\pi^4}{720}\Big(8a+8c+b+d\Big)\alpha'^2 + \\
&\quad + \Big(-\frac{1}{3}\zeta(5)(3a^2+3ac+3c^2+2b^2+2bd+2d^2+4ab+2ad+2bc+4cd) + \\
&\quad + \frac{\pi^2}{18}\zeta(3)(b^2+bd+d^2+2ab+ad+bc+2cd)\Big)\alpha'^3 + \mathcal{O}(\alpha'^4). \qquad \text{(C.56)}
\end{aligned}
$$

Using eqs. (C.11), (4.10) and (C.10), the α' factors defined in equations (C.49)–(C.53) can all be written in terms of the Gamma factor, Beta and Hypergeometric functions. It is also not difficult to see that these factors are all invariant under a twisting transformation with respect to index 1. The factor Δ in (C.54) is invariant under a twisting transformation with respect to any of the five indexes and it is also cyclic invariant.

Formulas (C.49), (C.50) and (C.53) can be understood to have no poles since the numerator is a power series of the α_{ij} and, whenever the denominator is zero, so happens with the numerator. This means that, in each of those cases, the numerator is divisible by the corresponding denominator.

In the case of formulas (C.51), (C.52) and (C.54) it is not straight forward to see that those expressions contain no poles, but the argument is similar: in all cases the numerator involves a power series of the α_{ij} and, whenever the denominator is zero, so happens with the numerator. For example, in the case of (C.51) and (C.52), using (C.19) (and one of the cyclic permutations of that relation) it can be proved that the numerator becomes zero if any of the α_{ij} of the denominator are zero. In the case of (C.54), using (C.20) and (C.21) (and the cyclic permutations of them) it can be proved that if any of the five α_{ij} of the denominator is zero, so happens with the numerator.

Up to $\mathcal{O}(\alpha'^6)$ order, the expansions of the factors defined in (C.49)–(C.54) are given by

$$
\begin{aligned}
H^{(1)} &= -\zeta(3)(2\alpha')^3 + \frac{\pi^4}{360}\Big[4\alpha_{45}+4\alpha_{23}+\alpha_{34}\Big](2\alpha')^4 + \\
&\quad + \Big[\frac{\pi^2}{6}\zeta(3)\alpha_{34}\Big(\alpha_{34}+\alpha_{45}+\alpha_{23}\Big) - \zeta(5)\times
\end{aligned}
$$

$$
\begin{aligned}
&\quad\times\left(2\alpha_{34}\alpha_{23}+\alpha_{23}^2+2\alpha_{45}\alpha_{34}+2\alpha_{34}^2+\alpha_{45}^2+\alpha_{45}\alpha_{23}\right)\Bigg](2\alpha')^5+\\
&+\Bigg[-\frac{1}{2}\zeta(3)^2\alpha_{34}\left(2\alpha_{23}\alpha_{34}+\alpha_{45}\alpha_{23}+2\alpha_{45}\alpha_{34}+\alpha_{45}^2+\alpha_{23}^2+\alpha_{34}^2\right)+\\
&\quad+\frac{\pi^6}{15120}\Big(16\alpha_{45}\alpha_{23}^2+12\alpha_{34}\alpha_{23}^2+12\alpha_{45}\alpha_{34}\alpha_{23}+16\alpha_{23}\alpha_{45}^2+23\alpha_{23}\alpha_{34}^2+\\
&\qquad\quad+23\alpha_{45}\alpha_{34}^2+16\alpha_{45}^3+12\alpha_{34}^3+16\alpha_{23}^3+12\alpha_{34}\alpha_{45}^2\Big)\Bigg](2\alpha')^6+\\
&+\mathcal{O}((2\alpha')^7)\,, \qquad\qquad (C.57)\\
P^{(1)}\;=\;&-\zeta(3)(2\alpha')^3+\frac{\pi^4}{360}\Big[4\alpha_{34}+\alpha_{45}+\alpha_{23}+4\alpha_{51}+4\alpha_{12}\Big](2\alpha')^4+\\
&+\Bigg[\frac{\pi^2}{6}\zeta(3)\Big(\alpha_{23}\alpha_{45}--2\alpha_{12}\alpha_{45}+\alpha_{45}^2+\alpha_{23}^2+\alpha_{34}\alpha_{45}+\alpha_{12}\alpha_{23}-\\
&\qquad\quad-2\alpha_{51}\alpha_{23}+\alpha_{23}\alpha_{34}+\alpha_{45}\alpha_{51}-2\alpha_{51}\alpha_{12}\Big)-\\
&\quad-\frac{1}{2}\zeta(5)\Big(2\alpha_{34}^2+2\alpha_{12}^2+4\alpha_{23}^2+2\alpha_{12}\alpha_{34}+2\alpha_{51}\alpha_{34}+2\alpha_{51}^2+4\alpha_{45}^2+4\alpha_{23}\alpha_{34}+\\
&\qquad\quad+4\alpha_{12}\alpha_{23}-7\alpha_{12}\alpha_{45}-7\alpha_{51}\alpha_{23}+4\alpha_{23}\alpha_{45}+4\alpha_{45}\alpha_{51}-7\alpha_{51}\alpha_{12}+\\
&\qquad\quad+4\alpha_{34}\alpha_{45}\Big)\Bigg](2\alpha')^5+\\
&+\Bigg[\frac{1}{2}\zeta(3)^2\Big(2\alpha_{51}^2\alpha_{23}+2\alpha_{12}^2\alpha_{45}-\alpha_{23}^2\alpha_{45}+\alpha_{23}^2\alpha_{51}-\alpha_{51}^2\alpha_{45}-\alpha_{34}^2\alpha_{45}-\alpha_{34}^2\alpha_{23}+\\
&\qquad\quad+\alpha_{12}^2\alpha_{51}-2\alpha_{12}\alpha_{23}^2-2\alpha_{45}^2\alpha_{51}-2\alpha_{23}^2\alpha_{34}-2\alpha_{45}^2\alpha_{34}+\alpha_{45}^2\alpha_{12}+\\
&\qquad\quad+\alpha_{51}^2\alpha_{12}-\alpha_{45}^2\alpha_{23}-\alpha_{12}^2\alpha_{23}-\alpha_{45}^3-\alpha_{23}^3+2\alpha_{12}\alpha_{34}\alpha_{51}-2\alpha_{23}\alpha_{34}\alpha_{45}+\\
&\qquad\quad+\alpha_{12}\alpha_{45}\alpha_{34}+4\alpha_{51}\alpha_{12}\alpha_{23}+\alpha_{51}\alpha_{23}\alpha_{34}+4\alpha_{12}\alpha_{23}\alpha_{45}+4\alpha_{45}\alpha_{51}\alpha_{23}-\\
&\qquad\quad-\alpha_{12}\alpha_{23}\alpha_{34}-\alpha_{34}\alpha_{45}\alpha_{51}+4\alpha_{45}\alpha_{51}\alpha_{12}\Big)+\frac{\pi^6}{45360}\times\\
&\quad\times\Big(-52\alpha_{51}^2\alpha_{23}-52\alpha_{12}^2\alpha_{45}+36\alpha_{23}^2\alpha_{45}-28\alpha_{23}^2\alpha_{51}+36\alpha_{51}^2\alpha_{45}\\
&\qquad+36\alpha_{34}^2\alpha_{45}+36\alpha_{34}^2\alpha_{23}-28\alpha_{12}^2\alpha_{51}+69\alpha_{12}\alpha_{23}^2+69\alpha_{45}^2\alpha_{51}+\\
&\qquad+69\alpha_{23}^2\alpha_{34}+69\alpha_{45}^2\alpha_{34}--28\alpha_{45}^2\alpha_{12}-28\alpha_{51}^2\alpha_{12}+36\alpha_{45}^2\alpha_{23}+36\alpha_{12}^2\alpha_{23}+\\
&\qquad+48\alpha_{34}^3+36\alpha_{45}^3+36\alpha_{23}^3+48\alpha_{51}^3+48\alpha_{12}^3+48\alpha_{34}\alpha_{51}^2-52\alpha_{12}\alpha_{34}\alpha_{51}+\\
&\qquad+69\alpha_{23}\alpha_{34}\alpha_{45}-28\alpha_{12}\alpha_{45}\alpha_{34}-134\alpha_{51}\alpha_{12}\alpha_{23}-28\alpha_{51}\alpha_{23}\alpha_{34}-\\
&\qquad-134\alpha_{12}\alpha_{23}\alpha_{45}-134\alpha_{45}\alpha_{51}\alpha_{23}+36\alpha_{12}\alpha_{23}\alpha_{34}+36\alpha_{34}\alpha_{45}\alpha_{51}-\\
&\qquad-134\alpha_{45}\alpha_{51}\alpha_{12}+48\alpha_{51}\alpha_{34}^2+48\alpha_{12}\alpha_{34}^2+48\alpha_{34}\alpha_{12}^2\Big)\Bigg](2\alpha')^6+\\
&+\mathcal{O}((2\alpha')^7)\,, \qquad\qquad (C.58)\\
U^{(1)}\;=\;&\frac{\pi^4}{90}(2\alpha')^4+\Bigg[-\frac{\pi^2}{3}\zeta(3)\Big(\alpha_{51}+\alpha_{12}\Big)+\frac{1}{2}\zeta(5)\Big(7\alpha_{51}+7\alpha_{12}-2\alpha_{45}-2\alpha_{23}\Big)\Bigg](2\alpha')^5+\\
&+\Bigg[\frac{1}{2}\zeta(3)^2\Big(4\alpha_{51}\alpha_{12}+\alpha_{51}^2+\alpha_{12}^2+2\alpha_{51}\alpha_{23}+2\alpha_{45}\alpha_{12}+\alpha_{45}\alpha_{51}+\alpha_{12}\alpha_{23}\Big)+\\
&\quad+\frac{\pi^6}{22680}\Big(-14\alpha_{45}\alpha_{51}+24\alpha_{23}^2-26\alpha_{45}\alpha_{12}-26\alpha_{51}\alpha_{23}+24\alpha_{45}^2-67\alpha_{51}\alpha_{12}-
\end{aligned}
$$

$$
- 14\alpha_{12}^2 + 24\alpha_{45}\alpha_{23} - 14\alpha_{51}^2 - 14\alpha_{12}\alpha_{23}\Big)\Big](2\alpha')^6 + \mathcal{O}((2\alpha')^7)\,, \qquad \text{(C.59)}
$$

$$
\begin{aligned}
W^{(1)} = {} & \frac{\pi^4}{360}(2\alpha')^4 + \Big[\frac{\pi^2}{6}\zeta(3)\Big(\alpha_{45} + \alpha_{23} - 2\alpha_{34} + \alpha_{51} + \alpha_{12}\Big) + \frac{1}{2}\zeta(5) \times \\
& \times \Big(7\alpha_{34} - 4\alpha_{45} - -4\alpha_{23} - 4\alpha_{51} - 4\alpha_{12}\Big)\Big](2\alpha')^5 + \\
& + \Big[-\frac{1}{2}\zeta(3)^2\Big(\alpha_{45}^2 + 2\alpha_{45}\alpha_{51} + \alpha_{51}^2 + \alpha_{12}^2 + \alpha_{23}^2 - 4\alpha_{23}\alpha_{45} + 2\alpha_{12}\alpha_{23} + \\
& + \alpha_{45}\alpha_{12} - 2\alpha_{34}^2 - \alpha_{34}\alpha_{51} + \alpha_{51}\alpha_{23} - 4\alpha_{23}\alpha_{34} - \alpha_{12}\alpha_{34} - \\
& - 4\alpha_{34}\alpha_{45} + 2\alpha_{51}\alpha_{12}\Big) + \frac{\pi^6}{45360} \times \\
& \times \Big(36\alpha_{51}\alpha_{23} - 28\alpha_{12}\alpha_{34} + 69\alpha_{51}\alpha_{12} + 36\alpha_{23}^2 + 36\alpha_{45}^2 + 69\alpha_{12}\alpha_{23} + 36\alpha_{51}^2 + \\
& + 36\alpha_{12}^2 - 134\alpha_{23}\alpha_{34} - 52\alpha_{34}^2 + 69\alpha_{45}\alpha_{51} - 134\alpha_{34}\alpha_{45} + \\
& + 36\alpha_{45}\alpha_{12} - 28\alpha_{34}\alpha_{51} - 134\alpha_{23}\alpha_{45}\Big)\Big](2\alpha')^6 + \mathcal{O}((2\alpha')^7)\,, \qquad \text{(C.60)}
\end{aligned}
$$

$$
\begin{aligned}
Z^{(1)} = {} & -\frac{\pi^4}{1440}(2\alpha')^4 + \Big(-\frac{\pi^2}{72}\zeta(3) + \frac{1}{6}\zeta(5)\Big)\Big(-\alpha_{12} + 3\alpha_{45} - \alpha_{51} + 6\alpha_{34} + 3\alpha_{23}\Big)(2\alpha')^5 + \\
& + \Big[\frac{1}{192}\zeta(3)^2\Big(21\alpha_{45}^2 + 6\alpha_{12}\alpha_{51} - 16\alpha_{23}\alpha_{51} + 3\alpha_{12}^2 + 3\alpha_{51}^2 + 21\alpha_{23}^2 + 96\alpha_{34}\alpha_{45} - \\
& - 16\alpha_{45}\alpha_{12} + 72\alpha_{34}^2 - 16\alpha_{23}\alpha_{12} - 24\alpha_{34}\alpha_{51} - 24\alpha_{34}\alpha_{12} - \\
& - 16\alpha_{45}\alpha_{51} + 96\alpha_{23}\alpha_{34} + +30\alpha_{45}\alpha_{23}\Big) - \frac{\pi^6}{362880} \times \\
& \times \Big(276\alpha_{23}\alpha_{34} - 72\alpha_{34}\alpha_{51} - 46\alpha_{45}\alpha_{51} + 90\alpha_{45}\alpha_{23} + 63\alpha_{23}^2 + +216\alpha_{34}^2 - \\
& - 46\alpha_{23}\alpha_{51} + 18\alpha_{12}\alpha_{51} + 9\alpha_{12}^2 + 9\alpha_{51}^2 + 63\alpha_{45}^2 - 46\alpha_{23}\alpha_{12} + 276\alpha_{34}\alpha_{45} - \\
& - 46\alpha_{45}\alpha_{12} - 72\alpha_{34}\alpha_{12}\Big)\Big](2\alpha')^6 + \mathcal{O}((2\alpha')^7)\,, \qquad \text{(C.61)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta = {} & \Big(\frac{3}{2}\pi^2\zeta(3) - \frac{35}{2}\zeta(5)\Big)(2\alpha')^5 + \Big(-\frac{3}{2}\zeta(3)^2 + \frac{109}{45360}\pi^6\Big) \times \\
& \times \Big(\alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{45} + \alpha_{51}\Big)(2\alpha')^6 + \mathcal{O}((2\alpha')^7)\,. \qquad \text{(C.62)}
\end{aligned}
$$

D. The Yang-Mills 5-point subamplitude

The calculation of the Yang-Mills 5-point subamplitude was already considered in the [34, appendix D].[30] Its (off-shell) expression is the following:

$$
\begin{aligned}
& A_{YM}(1,2,3,4,5) = \\
& = -ig^3 \times \\
& \times \Bigg\{ \frac{V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2) V_{YM}{}^{(3)\sigma}_{\mu_3\mu_4}(-k_3-k_4,k_3,k_4) V_{YM}{}^{(3)}_{\mu_5\rho\sigma}(k_5,k_1+k_2,k_3+k_4)}{(k_1+k_2)^2(k_3+k_4)^2} -
\end{aligned}
$$

[30] In [34] it was called as ‘5-point amplitude’.

$$- \frac{V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2) V_{YM}{}^{(4)}_{\rho\mu_3\mu_4\mu_5}(k_1+k_2,k_3,k_4,k_5)}{(k_1+k_2)^2} \Big\} \zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\zeta_5^{\mu_5} + $$
$$+ \Big(\text{cyclic permutations} \Big) , \tag{D.1}$$

where the Yang-Mills 3 and 4-point vertices (which do not carry color indices) are given, respectively, by

$$V^{(3)}_{YM\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = -i[\, \eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + \eta_{\mu_2\mu_3}(k_2-k_3)_{\mu_1} + \eta_{\mu_3\mu_1}(k_3-k_1)_{\mu_2} \,] , \tag{D.2}$$

$$V^{(4)}_{YM\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) = - \Big[\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - 2\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \eta_{\mu_4\mu_1}\eta_{\mu_2\mu_3} \Big] . \tag{D.3}$$

We have not worked any further trying to find a tensor notation which would shorten the expression of $A_{YM}(1,2,3,4,5)$ in (D.1).

E. The $D^{2n}F^4$ terms 5-point subamplitude

The 5-point subamplitude the comes from the lagrangian $\mathcal{L}_{D^{2n}F^4}$, given in (3.7), is given by:

$$\begin{aligned} &A_{D^{2n}F^4}(1,2,3,4,5) = \\ &= -2(2\alpha')^2 g^3 \Big\{ f(-2\alpha_{34},-2\alpha_{45}) \frac{T_{12}(\zeta,k)}{\alpha_{12}} + K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4) \times \\ &\qquad \times S_5(\zeta_5;k_1,k_2,k_3,k_4,k_5;\alpha') \Big\} + (\text{cyclic permutations}) , \end{aligned} \tag{E.1}$$

where

$$\begin{aligned} T_{12}(\zeta,k) = &\ \alpha_{12} K(\zeta_1,\zeta_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5) + (\zeta_1\cdot\zeta_2) K(k_1,k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5) + \\ &+ (\zeta_1\cdot k_2) K(\zeta_2,k_1+k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5) - \\ &- (\zeta_2\cdot k_1) K(\zeta_1,k_1+k_2;\zeta_3,k_3;\zeta_4,k_4;\zeta_5,k_5) \end{aligned} \tag{E.2}$$

and

$$\begin{aligned} &S_5(\zeta_5;k_1,k_2,k_3,k_4,k_5;\alpha') = \\ &= \Big[\Big(G(-2\alpha_{34},-2\alpha_{23},-\alpha_{12}-\alpha_{34},-\alpha_{14}-\alpha_{23}) + \\ &\qquad + G(-2\alpha_{12},-2\alpha_{23},-\alpha_{12}-\alpha_{34},-\alpha_{14}-\alpha_{23}) \Big) \zeta_5\cdot(k_1+k_2) + \\ &\quad + \Big(G(-2\alpha_{23},-2\alpha_{34},-\alpha_{14}-\alpha_{23},-\alpha_{12}-\alpha_{34}) - \\ &\qquad - G(-2\alpha_{23},-2\alpha_{12},-\alpha_{14}-\alpha_{23},-\alpha_{12}-\alpha_{34}) \Big) \zeta_5\cdot(k_1+k_4) \Big] . \end{aligned} \tag{E.3}$$

In these formulas, by $K(A,a;B,b;C,c;D,d)$ we mean the same expression of the 4-point amplitude kinematic factor, in (3.3), evaluated in the corresponding variables. The function

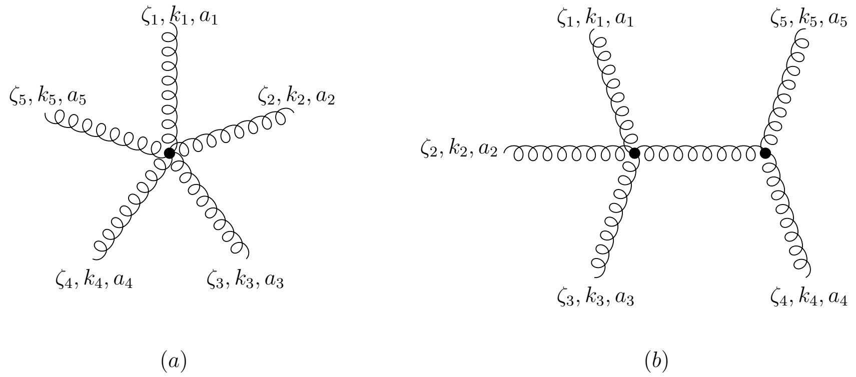

Figure 2: Type of 5-particle diagrams, with α' dependence, which come from the effective lagrangian. The Feynman diagram in (a) is one particle irreducible (1 PI), while the one in (b) is one particle reducible.

$f(s, t)$ in the second line of (E.1) is the one defined in (3.8) and the function $G(a, b, c, d)$ that goes in the second square bracket of (E.1) is the one defined in (C.55).

The derivation of our formula in (E.1) is quite nontrivial and very lengthy. We will not give all the details here, but will comment some aspects about it. It was obtained by first finding the 4 and 5-point vertices, $V^{(4)}_{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4)$ and $V^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5}(k_1, k_2, k_3, k_4, k_5)$ (which do not carry color indices), of the lagrangian in (3.7). This was done along the same lines of the Appendices of [17, 35], that is, by means of the 1-particle irreducible 4 and 5-point functions in momentum space.

In the case of the 1-particle irreducible 4-point function, $\Gamma^{(4)}_{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4)$, the calculation was straight forward since only the abelian terms of $\mathcal{L}_{D^{2n}F^4}$ were required.

In the case of the 1-particle irreducible 5-point function, $\Gamma^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5}(k_1, k_2, k_3, k_4, k_5)$, the calculation was much more involved than the previous one. On each $D^{2n}F^4$ term there are two types of 5-field terms present:

1. The ones that come from the non-abelian F^4 terms ($F^4 \to A^2 \cdot (\partial A)^3$), with the covariant derivatives acting as ordinary ones ($D^{2n} \to \partial^{2n}$).

   In the scattering amplitude (E.1) their contribution is the one that goes with the f factor, with no poles.

2. The ones that pick an A^a_μ term from the covariant derivatives ($D^{2n} \to A \cdot \partial^{2n-1}$) and the other four from the abelian part of the F^4 term ($F^4 \to (\partial A)^4$).

   These are the terms of the scattering amplitude (E.1) which are written in terms of the function $G(a, b, c, d)$. They begin to contribute only at order α'^3 (while the rest of the terms begin to contribute at α'^2 order) .

Once the 4 and 5-point vertices have been found, to all order in α', the calculation of the 5-point subamplitude in (E.1) can be done using the corresponding Feynman rules. This subamplitude receives contributions from one particle irreducible (1 PI) diagrams and one particle reducible ones which contain only simple poles, as shown in figure 2.

By construction, $A_{D^{2n}F^4}(1,2,3,4,5)$ remains invariant under cyclic permutations of indexes $(1,2,3,4,5)$. It is not difficult to see that it also remains invariant under a world-sheet parity (or twisting) transformation.

Since the lagrangian $\mathcal{L}_{D^{2n}F^4}$ is gauge invariant, $A_{D^{2n}F^4}(1,2,3,4,5)$ should also satisfy (on-shell) gauge invariance. This is a very non trivial test of formula (E.1). In fact, demanding that it should become zero after doing $\zeta_5 \to k_5$, it leads to the following relation to be satisfied:

$$
\begin{aligned}
& f(-2\alpha_{12},-2\alpha_{23}) - f(-2\alpha_{23},-2\alpha_{34}) = \\
& \quad = \Big[G(-2\alpha_{34},-2\alpha_{23},-\alpha_{12}-\alpha_{34},-\alpha_{14}-\alpha_{23}) + \\
& \qquad + G(-2\alpha_{12},-2\alpha_{23},-\alpha_{12}-\alpha_{34},-\alpha_{14}-\alpha_{23}) \Big] (\alpha_{51}+\alpha_{25}) + \\
& \qquad + \Big[G(-2\alpha_{23},-2\alpha_{34},-\alpha_{14}-\alpha_{23},-\alpha_{12}-\alpha_{34}) - \\
& \qquad - G(-2\alpha_{23},-2\alpha_{12},-\alpha_{14}-\alpha_{23},-\alpha_{12}-\alpha_{34}) \Big] (\alpha_{51}+\alpha_{45})\,. \qquad \text{(E.4)}
\end{aligned}
$$

We have checked this relation using the expansions of $f(s,t)$ and $G(a,b,c,d)$, given in (C.4) and (C.55), respectively, up to $\mathcal{O}(\alpha'^{15})$ terms.

F. Derivation of the $D^{2n}F^5$ terms 5-point subamplitude

Together with the derivation of formulas (4.13) and (E.1), the calculations of the present appendix constitute an essential part of this work. There are three main achievements that we arrive to in the expression for $A_{D^{2n}F^5}(1,2,3,4,5)$: it explicitly has no poles; it is written as a sum of terms that have manifest cyclic, (on-shell) gauge invariance and world-sheet parity symmetry; it is written in terms of tensors in such a way that a local lagrangian can be found almost directly in terms of them.

We have divided the calculations of this appendix in five steps:

Step 1. treatment of the poles of $T \cdot A_{YM}(1,2,3,4,5)$ in (4.13).

We begin writing the mentioned term as

$$
\begin{aligned}
& T \cdot A_{YM}(1,2,3,4,5) = \\
& \quad = A_{YM}(1,2,3,4,5) - ig^3 \times \\
& \qquad \times \Bigg\{ \left(T\Big|_{\alpha_{12}=0} - 1 \right) \times \\
& \qquad \times \Bigg[\frac{V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2) V_{YM}{}^{(3)\sigma}_{\mu_3\mu_4}(-k_3-k_4,k_3,k_4) V_{YM}{}^{(3)}_{\mu_5\rho\sigma}(k_5,k_1+k_2,k_3+k_4)}{4\alpha_{12}\alpha_{34}} - \\
& \qquad - \frac{V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2) V_{YM}{}^{(4)}_{\rho\mu_3\mu_4\mu_5}(k_1+k_2,k_3,k_4,k_5)}{2\alpha_{12}} \Bigg] \zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\zeta_5^{\mu_5} + \\
& \qquad + \text{(cyclic permutations)} \Bigg\} - ig^3 \times
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{ \left(T - T\Big|_{\alpha_{12}=0} \right) \times \right. \\
&\times \left[\frac{V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2)V_{YM}{}^{(3)\sigma}_{\mu_3\mu_4}(-k_3-k_4,k_3,k_4)V_{YM}{}^{(3)}_{\mu_5\rho\sigma}(k_5,k_1+k_2,k_3+k_4)}{4\alpha_{12}\alpha_{34}} - \right. \\
&\quad \left. - \frac{V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2)V_{YM}{}^{(4)}_{\rho\mu_3\mu_4\mu_5}(k_1+k_2,k_3,k_4,k_5)}{2\alpha_{12}} \right] \zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\zeta_5^{\mu_5} + \\
&\left. + \text{(cyclic permutations)} \right\}, \qquad \text{(F.1)}
\end{aligned}
$$

Here we have split the left-hand member of the equality into three terms on the right-hand side: $A_{YM}(1,2,3,4,5)$, the term which contains a factor $(T|_{\alpha_{12}=0}-1)$ (and cyclic permutations) and the term which contains a factor $(T-T|_{\alpha_{12}=0})$ (and cyclic permutations). We have used the on-shell expression of $A_{YM}(1,2,3,4,5)$ in these last two terms[31] and we have also used the fact that T is cyclic invariant.

In principle, $T \cdot A_{YM}(1,2,3,4,5)$ should have double poles because $A_{YM}(1,2,3,4,5)$ does, but it happens that the mentioned expression has only simple ones. To see this we notice that $(T-T|_{\alpha_{12}=0})$ is factorable by α_{12} and also, using (4.10) and (C.18), it may be proved that $(T|_{\alpha_{12}=0}-1)$ is factorable by $\alpha_{34}\alpha_{45}$. Using this information in (F.1) we can arrive to the following result:

$$
\begin{aligned}
&T \cdot A_{YM}(1,2,3,4,5) = \\
&\quad = A_{YM}(1,2,3,4,5) - ig^3 \left\{ \frac{1}{\alpha_{12}} \left[\left(\frac{T|_{\alpha_{12}=0}-1}{\alpha_{34}\alpha_{45}} \right) \times \right.\right. \\
&\qquad \times \left(\frac{\alpha_{45}}{4} V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2) V_{YM}{}^{(3)\sigma}_{\mu_3\mu_4} \times \right. \\
&\qquad\quad \times (-k_3-k_4,k_3,k_4) V_{YM}{}^{(3)}_{\mu_5\rho\sigma}(k_5,k_1+k_2,k_3+k_4) + \\
&\qquad\quad + \frac{\alpha_{34}\alpha_{45}}{2} V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2) \times \\
&\qquad\quad \left. \times V_{YM}{}^{(4)}_{\rho\mu_3\mu_4\mu_5}(k_1+k_2,k_3,k_4,k_5) \right) + \left(\frac{T-T|_{\alpha_{45}=0}}{\alpha_{45}} \right) \times \\
&\qquad \times \frac{1}{4} V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2) V_{YM}{}^{(3)\sigma}_{\mu_3\mu_4} \times \\
&\qquad \left. \times (-k_3-k_4,k_3,k_4) V_{YM}{}^{(3)}_{\mu_5\rho\sigma}(k_5,k_1+k_2,k_3+k_4) \right] \times \\
&\qquad \left. \times \zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\zeta_5^{\mu_5} + \text{(cyclic permutations)} \right\} - \\
&\quad - ig^3 \left\{ \left(\frac{T-T|_{\alpha_{12}=0}}{\alpha_{12}} \right) \frac{1}{2} V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2) V_{YM}{}^{(4)}_{\rho\mu_3\mu_4\mu_5}(k_1+k_2,k_3,k_4,k_5) \times \right. \\
&\qquad \left. \times \zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\zeta_5^{\mu_5} + \text{(cyclic permutations)} \right\}, \qquad \text{(F.2)}
\end{aligned}
$$

in which is manifest that $T \cdot A_{YM}(1,2,3,4,5)$ contains only simple poles: they come in the first curly bracket of this equation.

[31] This has been done by substituting $(k_i+k_j)^2 = 2\alpha_{ij}$ in the denominators of (D.1).

Step 2. Treatment of the poles of $(2\alpha')^2 K_3 \cdot A_{F^4}(1,2,3,4,5)$ in (4.13).

In a similar way as it was done in Step 1, using the expression in (4.14) for $A_{F^4}(1,2,3,4,5)$, we arrive to:

$$
\begin{aligned}
&(2\alpha')^2 K_3 \cdot A_{F^4}(1,2,3,4,5) = \\
&\quad = -2g^3 \left\{ (2\alpha')^2 f(-2\alpha_{34}, -2\alpha_{45}) \frac{T_{12}(\zeta,k)}{\alpha_{12}} + \Big(\text{cyclic permutations} \Big) \right\} + \\
&\qquad + 2g^3 \left\{ \left(\frac{T|_{\alpha_{12}=0} - 1}{\alpha_{34}\alpha_{45}} \right) \frac{T_{12}(\zeta,k)}{\alpha_{12}} + \Big(\text{cyclic permutations} \Big) \right\} + \\
&\qquad + 2g^3 \left\{ (2\alpha')^2 \left(\frac{K_3 - K_3|_{\alpha_{12}=0}}{\alpha_{12}} \right) T_{12}(\zeta,k) + \Big(\text{cyclic permutations} \Big) \right\},
\end{aligned}
\tag{F.3}
$$

where $T_{12}(\zeta,k)$ is given in (E.2).

The simple poles appear in the first and second lines of (F.3). In the second line of this equation we used that

$$
(2\alpha')^2 K_3 \Big|_{\alpha_{12}=0} + (2\alpha')^2 f(-2\alpha_{34}, -2\alpha_{45}) = \frac{T|_{\alpha_{12}=0} - 1}{\alpha_{34}\alpha_{45}}\,,
\tag{F.4}
$$

which is equivalent to (C.17), once the double pole of that equation has been subtracted in both sides of it.

We notice that the simple poles in the first line of (F.3) are *exactly the same ones* of $A_{D^{2n}F^4}(1,2,3,4,5)$ (see eq. (E.1)).

Step 3. Taking the poles to $A_{D^{2n}F^4}(1,2,3,4,5)$.

Using the expressions in (F.2) and (F.3) we have that

$$
\begin{aligned}
&T \cdot A_{YM}(1,2,3,4,5) + (2\alpha')^2 K_3 \cdot A_{F^4}(1,2,3,4,5) = \\
&\quad = A_{YM}(1,2,3,4,5) - 2g^3 \left\{ (2\alpha')^2 f(-2\alpha_{34}, -2\alpha_{45}) \frac{T_{12}(\zeta,k)}{\alpha_{12}} + (\text{cyclic permutations}) \right\} + \\
&\qquad + g^3 \left\{ \frac{1}{\alpha_{12}} \left[\left(\frac{T|_{\alpha_{12}=0} - 1}{\alpha_{34}\alpha_{45}} \right) \times \right.\right. \\
&\qquad\qquad \times \left(2T_{12}(\zeta,k) - -i \frac{\alpha_{45}}{4} V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1, k_2, -k_1-k_2) V_{YM}{}^{(3)\sigma}_{\mu_3\mu_4} \times \right. \\
&\qquad\qquad\quad \times (-k_3-k_4, k_3, k_4) V_{YM}{}^{(3)}_{\mu_5\rho\sigma}(k_5, k_1+k_2, k_3+k_4) + \\
&\qquad\qquad\quad \left. + i \frac{\alpha_{34}\alpha_{45}}{2} V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1, k_2, -k_1-k_2) V_{YM}{}^{(4)}_{\rho\mu_3\mu_4\mu_5}(k_1+k_2, k_3, k_4, k_5) \right) - \\
&\qquad\qquad - \left(\frac{T - T|_{\alpha_{45}=0}}{\alpha_{45}} \right) \frac{i}{4} V_{YM}{}^{(3)\rho}_{\mu_4\mu_5}(k_4, k_5, -k_4-k_5) V_{YM}{}^{(3)\sigma}_{\mu_1\mu_2} \times \\
&\qquad\qquad \left. \times (-k_1-k_2, k_1, k_2) V_{YM}{}^{(3)}_{\mu_3\rho\sigma}(k_3, k_4+k_5, k_1+k_2) \right] \times \\
&\qquad \left. \times \zeta_1^{\mu_1} \zeta_2^{\mu_2} \zeta_3^{\mu_3} \zeta_4^{\mu_4} \zeta_5^{\mu_5} + \Big(\text{cyclic permutations} \Big) \right\} - \\
&\quad - i g^3 \left\{ \left(\frac{T - T|_{\alpha_{12}=0}}{\alpha_{12}} \right) \frac{1}{2} V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1, k_2, -k_1-k_2) V_{YM}{}^{(4)}_{\rho\mu_3\mu_4\mu_5}(k_1+k_2, k_3, k_4, k_5) \times \right.
\end{aligned}
$$

$$\times \zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\zeta_5^{\mu_5} \quad + \quad \Big(\text{cyclic permutations}\Big)\Bigg\} +$$

$$+\,2g^3\left\{(2\alpha')^2\left(\frac{K_3 - K_3|_{\alpha_{12}=0}}{\alpha_{12}}\right)T_{12}(\zeta,k) + (\text{cyclic permutations})\right\}. \tag{F.5}$$

Now, we first notice that

$$\left(\frac{T - T|_{\alpha_{45}=0}}{\alpha_{45}}\right) - \alpha_{34}\left(\frac{T|_{\alpha_{12}=0} - 1}{\alpha_{34}\alpha_{45}}\right) \tag{F.6}$$

is factorable by α_{12},[32] so we will call it $\alpha_{12}M_{12}$, where M_{12} has a well defined α' series expansion with no poles. Eliminating $(T - T|_{\alpha_{45}=0})/\alpha_{45}$ from it and substituting it in (F.5), together with the expression of $A_{D^{2n}F^4}(1,2,3,4,5)$, given in (E.1), we arrive to

$$\begin{aligned}
&T \cdot A_{YM}(1,2,3,4,5) + (2\alpha')^2 K_3 \cdot A_{F^4}(1,2,3,4,5) = \\
&\quad = A_{YM}(1,2,3,4,5) + A_{D^{2n}F^4}(1,2,3,4,5) + g^3 \times \\
&\quad\times\left\{\left(\frac{T|_{\alpha_{12}=0} - 1}{\alpha_{34}\alpha_{45}}\right) \times \left(\frac{1}{\alpha_{12}}\Big[2T_{12}(\zeta,k) - i\frac{\alpha_{45}}{4}V_{YM}{}^{(3)\rho}_{\mu_1\mu_2}(k_1,k_2,-k_1-k_2)V_{YM}{}^{(3)\sigma}_{\mu_3\mu_4}\times\right.\right. \\
&\qquad\times(-k_3-k_4,k_3,k_4)V_{YM}{}^{(3)}_{\mu_5\rho\sigma}(k_5,k_1+k_2,k_3+k_4) - \\
&\qquad - i\frac{\alpha_{34}}{4}V_{YM}{}^{(3)\rho}_{\mu_4\mu_5}(k_4,k_5,-k_4-k_5)V_{YM}{}^{(3)\sigma}_{\mu_1\mu_2}\times \\
&\qquad\times(-k_1-k_2,k_1,k_2)V_{YM}{}^{(3)}_{\mu_3\rho\sigma}(k_3,k_4+k_5,k_1+k_2) + \\
&\qquad + i\frac{\alpha_{34}\alpha_{45}}{2}V_{YM}{}^{(3)}_{\mu_1\mu_2}{}^{\rho}(k_1,k_2,-k_1-k_2)\times \\
&\qquad\times V_{YM}{}^{(4)}_{\rho\mu_3\mu_4\mu_5}(k_1+k_2,k_3,k_4,k_5)\Big] + \\
&\qquad + i\frac{\alpha_{34}}{2}V_{YM}{}^{(3)\rho}_{\mu_4\mu_5}(k_4,k_5,-k_4-k_5)V_{YM}{}^{(4)}_{\rho\mu_1\mu_2\mu_3}\times \\
&\qquad\left.\times(k_4+k_5,k_1,k_2,k_3)\right)\zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\zeta_5^{\mu_5} + \\
&\quad\left. + (\text{cyclic permutations})\right\} - ig^3 \times \\
&\quad\times\left\{M_{12}\left(\frac{1}{4}V_{YM}{}^{(3)\rho}_{\mu_4\mu_5}(k_4,k_5,-k_4-k_5)V_{YM}{}^{(3)\sigma}_{\mu_1\mu_2}(-k_1-k_2,k_1,k_2)V_{YM}{}^{(3)}_{\mu_3\rho\sigma}\times\right.\right. \\
&\qquad\times(k_3,k_4+k_5,k_1+k_2) - \frac{\alpha_{12}}{2}V_{YM}{}^{(3)\rho}_{\mu_4\mu_5}(k_4,k_5,-k_4-k_5)\times \\
&\qquad\left.\times V_{YM}{}^{(4)}_{\rho\mu_1\mu_2\mu_3}(k_4+k_5,k_1,k_2,k_3)\right)\zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\zeta_5^{\mu_5} + \\
&\quad\left. + (\text{cyclic permutations})\right\} + \\
&\quad + 2g^3\left\{(2\alpha')^2\left(\frac{K_3 - K_3|_{\alpha_{12}=0}}{\alpha_{12}}\right)T_{12}(\zeta,k) + \Big(\text{cyclic permutations}\Big)\right\} + \\
&\quad + 2g^3(2\alpha')^2\Big\{K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4)S_5(\zeta_5;k_1,k_2,k_3,k_4,k_5;\alpha') +
\end{aligned}$$

[32] The expression in (F.6) has a power series expansion in α_{12} and using (one of the cyclic permutations of) (C.21) it can be proved that it becomes 0 when $\alpha_{12} = 0$.

$$+\left(\text{ cyclic permutations }\right)\Bigg\}. \tag{F.7}$$

This is a huge expression. At this moment our main interest lies in the term in the square bracket (the one that goes multiplying the factor $1/\alpha_{12}$) and its cyclic permutations because, besides the poles of $A_{D^{2n}F^4}(1,2,3,4,5)$, those are the only places where poles could come from. We have verified computationally that, using on-shell and physical state conditions, together with momentum conservation, the term in the square bracket is factorable by α_{12}, thus eliminating all poles which are not contained in $A_{D^{2n}F^4}(1,2,3,4,5)$.

Step 4. Derivation of an expression for $A_{D^{2n}F^5}(1,2,3,4,5)$ as a sum of terms which are gauge invariant and have no poles.

Since the left hand-side of (F.7) is precisely $A(1,2,3,4,5)$, using (5.1) we have that the sum of curly brackets in (F.7) corresponds to $A_{D^{2n}F^5}(1,2,3,4,5)$. This expression explicitly has no poles and is (on-shell) gauge invariant as a whole, but each of the terms in curly brackets is not individually gauge invariant. This means that, in order to write $A_{D^{2n}F^5}(1,2,3,4,5)$ as a sum of terms which have both properties, some redistribution of the terms in (F.7) should be done. Only after succeeding in doing this redistribution it will be possible to find a local lagrangian for each group of terms.

This is a very non trivial step and in the following lines we will only summarize the operations which took us to the desired expression:

1. In (F.7) we substitute the following expression (and the cyclic permutations of it in the corresponding cases) for the kinematical expression $T_{12}(\zeta,k)$, defined in (E.2):

$$\begin{aligned} T_{12}(\zeta,k) =\, & (\eta\cdot t_{(8)})_4{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\zeta^1_{\mu_1}k^1_{\nu_1}\zeta^2_{\mu_2}k^2_{\nu_2}\zeta^3_{\mu_3}k^3_{\nu_3}\zeta^4_{\mu_4}k^4_{\nu_4}\zeta^5_{\mu_5}k^5_{\nu_5}+\\ &+(\zeta^1\cdot k^2)K(\zeta^2,k^2;\zeta^3,k^3;\zeta^4,k^4;\zeta^5,k^5)-\\ &-(\zeta^2\cdot k^1)K(\zeta^1,k^1;\zeta^3,k^3;\zeta^4,k^4;\zeta^5,k^5)\,. \end{aligned} \tag{F.8}$$

   The term with the $(\eta\cdot t_{(8)})_4$ tensor is the part of $T_{12}(\zeta,k)$ which is gauge invariant. That term changes sign under a twisting transformation with respect to index 4.

2. In the term $S_5(\zeta_5;k_1,k_2,k_3,k_4,k_5;\alpha')$ given in (E.3), which also goes in (F.7), we substitute the following relation (and the corresponding cyclic permutations of it):

$$\begin{aligned} & G(-2\alpha_{34},-2\alpha_{23},-\alpha_{12}-\alpha_{34},-\alpha_{14}-\alpha_{23})+\\ &+G(-2\alpha_{12},-2\alpha_{23},-\alpha_{12}-\alpha_{34},-\alpha_{14}-\alpha_{23})=\\ &=\frac{f(-2\alpha_{12},-2\alpha_{23})-f(-2\alpha_{23},-2\alpha_{34})}{\alpha_{34}-\alpha_{12}}-\\ &\quad-\Big[G(-2\alpha_{23},-2\alpha_{34},-\alpha_{14}-\alpha_{23},-\alpha_{12}-\alpha_{34})-\\ &\qquad-G(-2\alpha_{23},-2\alpha_{12},-\alpha_{14}-\alpha_{23},-\alpha_{12}-\alpha_{34})\Big]\left(\frac{\alpha_{51}+\alpha_{45}}{\alpha_{34}-\alpha_{12}}\right). \end{aligned} \tag{F.9}$$

   This relation has been obtained from (E.4) and in the denominators we have used, from (C.15), that $\alpha_{51}+\alpha_{25}=\alpha_{34}-\alpha_{12}$.

3. We then substitute in (F.7) the following expressions for M_{12} and $(T|_{\alpha_{12}=0}-1)/(\alpha_{34}\alpha_{45})$ (and the corresponding cyclic permutations of them):

$$M_{12} = \alpha_{51}H^{(3)} + \alpha_{51}\alpha_{23}U^{(1)} + \alpha_{23}\alpha_{34}U^{(3)} + \alpha_{34}\alpha_{51}U^{(5)} + \alpha_{34}\alpha_{51}\alpha_{23}\Delta\,, \tag{F.10}$$

$$\frac{T|_{\alpha_{12}=0}-1}{\alpha_{34}\alpha_{45}} = \alpha_{23}H^{(1)} + \alpha_{51}H^{(2)} + \alpha_{23}\alpha_{51}U^{(4)}\,. \tag{F.11}$$

These relations have been obtained from the definition of M_{12}, given in (F.6), and using the definitions of the $H^{(k)}$, $U^{(k)}$ and Δ factors of appendix C.3, together with (F.4).

As an outcome, the resulting expression of $A_{D^{2n}F^5}(1,2,3,4,5)$ will depend on kinematical expressions and on the factors $H^{(k)}$, $P^{(k)}$, $U^{(k)}$, $Z^{(k)}$ $(k=1,\ldots,5.)$ and Δ. It will no longer depend (explicitly) on the T and K_3 factors.

4. We next require on-shell gauge invariance in the resulting expression of $A_{D^{2n}F^5}(1,2,3,4,5)$ by doing, for example, $\zeta_1 \to k_1$ and demanding that the expression should vanish after using on-shell and physical state conditions, together with momentum conservation. This leads us to the following condition to be satisfied:

$$\alpha_{51}P^{(3)} - \alpha_{12}P^{(4)} = \alpha_{51}H^{(2)} - \alpha_{12}H^{(5)} + \alpha_{23}\alpha_{51}U^{(4)} - \alpha_{12}\alpha_{45}U^{(3)}\,, \tag{F.12}$$

which can be seen to be valid after using the definitions of the $H^{(k)}$, $P^{(k)}$ and $U^{(k)}$ factors, given in appendix C.3.

In order for the condition (F.12) to be automatically satisfied, without needing to make any substitutions, we introduce a new α' dependent factor, $W^{(1)}$, and its cyclic permutations $W^{(k)}$ $(k=2,3,4,5)$. This factor has been defined in eq. (C.52) and it may be proved that it satisfies the following relations:

$$P^{(4)} = H^{(5)} + \alpha_{45}U^{(3)} + \alpha_{51}W^{(1)}\,, \tag{F.13}$$

$$P^{(3)} = H^{(2)} + \alpha_{23}U^{(4)} + \alpha_{12}W^{(1)}\,, \tag{F.14}$$

which automatically fulfill (F.12).

Step 5. Final expression for $A_{D^{2n}F^5}(1,2,3,4,5)$.

From the previous step we find the following expression for $A_{D^{2n}F^5}(1,2,3,4,5)$:

$$\begin{aligned} A_{D^{2n}F^5}(1,2,3,4,5) &= \\ = g^3\Bigg\{\Bigg[&H^{(1)}\cdot h(\zeta,k) + P^{(1)}\cdot p^{(1)}(\zeta,k) + +U^{(1)}\cdot u(\zeta,k) + W^{(1)}\cdot w(\zeta,k) + \\ &+ Z^{(1)}\cdot z^{(1)}(\zeta,k)\Bigg] + (\text{cyclic permutations})\Bigg\} + g^3\Delta\cdot\delta'(\zeta,k)\,, \end{aligned} \tag{F.15}$$

where, by now, only $p^{(1)}(\zeta,k)$ and $z^{(1)}(\zeta,k)$ (and their cyclic permutations) are known to be the same kinematical expressions of equations (5.4) and (5.7), respectively.

In (F.15) we have an expression which, on *each* group of terms, has no poles and is (on-shell) gauge invariant. We have also verified that the world-sheet parity condition (4.16) is satisfied by *each* group of terms in (F.15). But unfortunately, for $h(\zeta, k)$, $u(\zeta, k)$, $w(\zeta, k)$ and $\delta'(\zeta, k)$ we only have long (computer saved) expressions, whose specific structure is not explicitly known.

Our final labor has then been to determine the structure of $h(\zeta, k)$, $u(\zeta, k)$, $w(\zeta, k)$ and $\delta'(\zeta, k)$. In appendix B.2 we have explained with some detail how we have obtained an on-shell equivalent expression for $h(\zeta, k)$ (which we have called $h^{(1)}(\zeta, k)$ in equations (5.2) and (5.3)) in which the gauge symmetry is manifest. This has been done by introducing a ten index tensor, $t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}$, which is antisymmetric on each pair $(\mu_j\nu_j)$ and which has the twisting symmetry mentioned in eq. (B.3).

For each remaining kinematical expression we have also used an *ansatz*, in analogy to $h^{(1)}(\zeta, k)$, which consists in the most general expression that can be constructed and which is manifestly gauge invariant. It is quite remarkable that using on-shell, physical state and momentum conservation conditions, we have been able to determine their structure completely in terms of only two tensors: the known $t_{(8)}$ and the same $t_{(10)}$ that we have mentioned in the above lines. The resulting expressions are the ones that we have named $u^{(1)}(\zeta, k)$, $w^{(1)}(\zeta, k)$ and $\delta(\zeta, k)$, respectively, in subsection 5.1.

The final expression that we have for $A_{D^{2n}F^5}(1, 2, 3, 4, 5)$ is, then, the one in (5.2), which has no poles, is cyclic and (on-shell) gauge invariant on *each* group of terms, as explained in subsection 5.1, and which also has the world-sheet parity symmetry manifest. All this final step has been fundamental, in order to go from the subamplitude in (5.2) to the effective lagrangian, $\mathcal{L}_{D^{2n}F^5}$, in (5.9).

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